

Total Degree Formula for the Generic Offset to a Parametric Surface*

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Introduction

This paper focuses on the study of the total degree w.r.t. the spatial variables of the multivariate polynomial defining the generic offset to a rational surface in three-dimensional space. So, before continuing with this introduction, let us describe, at least informally, the offsetting construction; for a detailed explanation see Section 1 and, more specifically, for the concept of generic offset, see Definition 1.13 (page 16).

Let Σ be a surface in three-dimensional space. At each point \bar{p} of Σ , consider the normal line \mathcal{L}_Σ to the surface (assume, for this informal introduction, that the normal

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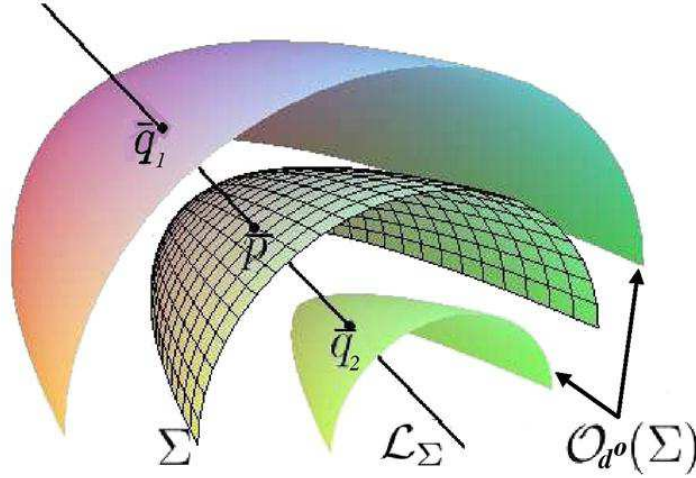


Figure 1: Informal Definition of Offset to a Generating Surface

line to Σ at \bar{p} is well defined). Let \bar{q} be a point on that line, at a distance d^o of \bar{p} (there are two such points \bar{q}); equivalently, we consider the intersection points of \mathcal{L}_Σ with a sphere centered at \bar{p} and with radius d^o . The offset surface to Σ at distance value d^o , is the set $\mathcal{O}_{d^o}(\Sigma)$ of all the points \bar{q} obtained by this geometric construction, illustrated in Figure 1. Σ is said to be the **generating surface** of $\mathcal{O}_{d^o}(\Sigma)$.

The classical offset construction for algebraic hypersurfaces has been, and still is, an active research subject of scientific interest. Even though the historical origins of the study of offset curves can be traced back to the work of classical geometers ([19],[20],[31]), often under the denomination of *parallel curves*, the subject received increased attention when the technological advance in the fields of Computer Assisted Design and Computer Assisted Manufacturing (CAD/CAM) resulted in a strong demand of effective algorithms for the manipulation of curves and surfaces. We quote the following from one of the two seminal papers ([13],[14]) by Farouki and Neff: “*Apart from numerical-control machining, offset curves arise in a variety of practical applications such as tolerance analysis, geometric optics, robot path-planning, and in the formulation of simple geometric procedures (growing/shrinking, blending, filleting, etc.)*”. To the applications listed by these authors we should add here some recent ones, e.g. the connection with the medial axis transform; for these, and related applications see Chapter 11 in [23], and the references contained therein.

As a result of this interest coming from the applications, many new methods and algorithms have been developed by engineers and mathematicians, and many geometric and algebraic properties of the offset construction have been studied in recent years; see, e.g. the references [2],[3],[4],[5],[6],[7],[8],[13],[14],[17],[18],[21],

[22],[27],[28],[29],[33],[35],[38],[39]. In addition to these references, we also refer to the theses [1], [32] and [37], developed within the research group of Prof. J.R. Sendra. In [37] the fundamental algebraic properties of offsets to hypersurfaces are deduced, the unirationality of the offset components are characterized, and the genus problem (for the curve case) is studied. In [1] the topological behavior of the offset curve is analyzed. The present paper contains the results about the total degree of the generic offset to a rational surface in Chapter 4 of the Ph.D. Thesis [32] .

With the exception of certain degenerated situations, that are indeed well known, the offset to an algebraic hypersurface is again a hypersurface (see [38]). Thus, one might answer all the problems mentioned above (parametrization expressions, genus computation, topologic types determination, degree analysis, etc), by applying the available algorithms to the resulting (offset) hypersurface. However, in most cases, this strategy results unfeasible. The reason is that the offsetting process generates a huge size increment of the data defining the offset in comparison to the data of the original variety. The challenge, therefore, is to derive information (say algebraic or geometric properties) of the offset hypersurface from the information that could be easily derived from the original (in general much simpler) hypersurface.

Framed in the above philosophy, the goal of this paper is to provide an efficient formula for the total degree of the generic offset to a rational surface. More precisely: let $f(y_1, y_2, y_3)$ be the defining polynomial of Σ , and let us treat d as variable. Then, we introduce a new polynomial $g(d, x_1, x_2, x_3)$ such that for almost all non-zero values d^o of d the specialization $g(d^o, x_1, x_2, x_3)$ defines the offset to Σ at distance d^o . Such a polynomial is called the generic offset polynomial (see Definition 1.16, page 18), and the hypersurface that it defines (in four-dimensional space) is called the generic offset of Σ (see Def 1.13, page 16). In this situation, the goal of this paper is to describe an effective solution for the problem of computing the total degree in $\{x_1, x_2, x_3\}$ of g .

There exist some (few) contributions in the literature concerning the degree problem for offset curves and surfaces. To our knowledge, the first attempt to provide a degree formula for offset curves was given by Salmon, in [31]. This formula was proved wrong in the already mentioned paper by Farouki and Neff [13]. In this paper, the authors provide a degree formula for rational curves given parametrically. They also deal separately with the case of polynomial parametrizations. Our papers [33] and [35] provide a complete and efficient solution for the problem of computing the degree structure (that is total and partial degrees, and degree w.r.t. the distance) for the case of plane algebraic curves, both in the implicit and the parametric case. Finally, let us mention that Anton et al. provide in [6] an alternative formula (to those presented in [33]) for computing the total degree of the offsets to an algebraic curve.

In contrast with the case of curves, even in the case of a generating parametric surface, there are, up to our knowledge, no available results for the offset degree problem in the scientific literature. In this paper, concretely in Theorem 3.22 (page 73) we provide a

formula for the total degree of the generic offset to a rational surface, given in parametric form, provided that the Assumption 2.6 holds (see page 35). **The parametrization of the surface is not assumed to be proper**, and the formula in fact provides the product of the total offset degree times the tracing index of the parametrization. However, since there are available efficient algorithms for computing the tracing index of a surface parametrization (see [26]) this assumption does not limit the applicability of the formula.

The strategy for this offset degree problem is, as in our previous papers, based in the analysis of the intersection between the generic offset and a pencil of lines through the origin. The restriction to the rational case, combined with this strategy, results in a reduction in the dimension of the space needed to study of the intersection problem. Thus, we are led to consider again an intersection problem of plane curves. The auxiliary curves involved in this case are obtained by eliminating the variables corresponding to a point in the generating surface from the offset-line intersection system. The main technical differences between this paper and our previous ones (see [33] and [35]) are that:

- Here we need to consider more than two intersection curves. Thus, the total degree formula is expressed as a generalized resultant of the equations of these auxiliary curves.
- Furthermore, all the curves involved in the intersection problem depend on parameters. Thus, the notion of fake point and their characterization is technically more demanding.

Generally speaking, the dimensional advantage gained by working with a parametric representation is partially compensated by the fact that we are not dealing directly with the points of the surface but with their parametric representation, and thus we are losing some geometric intuition. In the general situation of an implicitly given generating surface, if one were to apply a similar strategy to the offset degree problem, we believe that one is bound to consider a surface intersection problem, instead of the simpler curve intersection problem used here. However, in this paper we do not address the offset degree problem in that general situation.

As those skilled in the art know, going from the curve to the surface case usually implies a huge step in the difficulty of the proofs. For us, this has indeed been the case. As a result, some of the proofs in this paper are rather technical. And in one particular case, we have not been able to extend to the surface case the proof of a result that we obtained for plane curves. Specifically, in Lemma 4 of our paper [33] we proved that there are only finitely many distance values d^o for which the origin belongs to $\mathcal{O}_{d^o}(\mathcal{C})$. Our conjecture is that a similar property holds for all algebraic surfaces. However, as we said, we have not been able to provide a proof. The following example shows that, even for curves, this property does not hold if we consider the analytic case

Example. We want to emphasize that this proposition does not hold in a non-algebraic context. For example, the “offset” to the analytic curve with implicit equation $y^3 - \sin(x) = 0$, passes through the origin for infinitely many values of d . In fact, for this curve, all the offsets with values of d equal to $k\pi$, for $k \in \mathbb{Z}$, pass through the origin. This is illustrated in Figure 2, where the curve (in red) and some offsets passing through the origin (for $k = 1, 2, 3$) are shown.

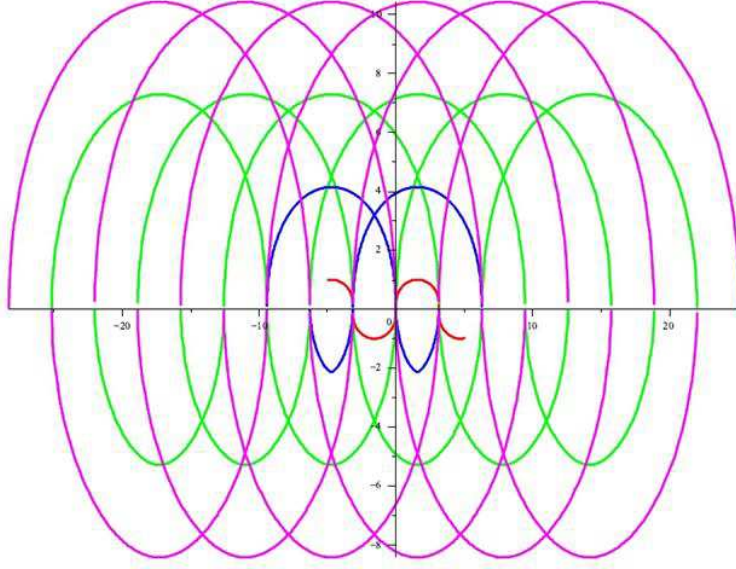


Figure 2: A smooth curve with infinitely many offsets through the origin

Besides, because of its own nature, our strategy fails in the case of some simple surfaces. We have met similar situations in our previous papers ([33] and [35]), where we needed to exclude circles centered at the origin and lines through the origin from our considerations. Correspondingly, in this paper we need to exclude the case in which the generating surface is a sphere centered at the origin. In this case, however, the generic offset degree (in fact the generic offset equation) is known beforehand. Therefore, excluding it does not really affect the generality of the degree formula that we present here. The above observations are the reason for the following assumptions:

Assumptions 2.6 (page 35) Let Σ denote the generating surface. In this paper, we assume that:

- (1) There exists a finite subset Δ^1 of \mathbb{C} such that, for $d^o \notin \Delta^1$ the origin does not belong to $\mathcal{O}_{d^o}(\Sigma)$.

- (2) Σ is not a sphere centered at the origin.

Structure of the Paper

In Subsection 1.1 of **Section 1** (page 10) we begin by reviewing the fundamental concepts related to the classical offset construction, and its basic properties that will be used in the sequel. Then we introduce the notion of generic offset, which can be considered as a hypersurface that collects as level curves all the classical offsets to a given hypersurface. This notion is the natural generalization of the classical concept, by considering the distance as a new variable. The defining polynomial of this object is the generic offset polynomial; see Definition 1.16 (page 18). We establish the fundamental specialization property of the generic offset polynomial in Theorem 1.19 (page 19). In Subsection 1.2, we recall some basic notions on parametric algebraic surfaces, and some technical lemmas about them. We introduce the notion of associated normal vector, and we also construct a parametric analogous of the Generic Offset System. In Subsection 1.3 (page 27) we present two technical lemmas about the use of univariate resultants to study the problem of the intersection of plane algebraic curves.

Section 2 (page 30) describes the theoretical foundation of the strategy. Subsection 2.1 contains the analysis of the intersection between the generic offset and a pencil of lines through the origin. In Subsection 2.2 we will see that, when elimination techniques are brought into our strategy, the dimension of the space in which we count the points in $\mathcal{O}_d(\Sigma) \cap \mathcal{L}_{\bar{k}}$ is reduced, and we arrive at an intersection problem between projective plane curves. Then we begin the analysis of that problem. Specifically, in Subsection 2.2 we describe the auxiliary polynomials obtained by using elimination techniques in the Parametric Offset-Line System, and we introduce the Auxiliary System 12 (page 38), denoted by $\mathfrak{S}_3^P(d, \bar{k})$. Some geometric properties of the solutions of $\mathfrak{S}_3^P(d, \bar{k})$ (see Proposition 2.11, page 40 and Lemma 2.13, page 43) will be used in the sequel to study the relation between the solution sets of Systems $\mathfrak{S}_2^P(d, \bar{k})$ and $\mathfrak{S}_3^P(d, \bar{k})$. In Subsection 2.3 (page 44) we define the corresponding notion of fake points and invariant points for the Affine Auxiliary System $\mathfrak{S}_3^P(d, \bar{k})$. The relation between these two notions is then shown in Proposition 2.18 (page 46).

The statement and proof of the degree formula appear in **Section 3** (page 47). This section is structured into four subsections as follows. In Subsection 3.1 we study the projective version of the auxiliary curves introduced in the preceding section, and we introduce the Projective Auxiliary System 30 (page 52). The polynomials that define this system are the basic ingredients of the degree formula. Subsection 3.2. (page 52) deals with the invariant solutions of the Projective Auxiliary System. In Subsection 3.3 (page 61) we will prove that the value of the multiplicity of intersection of the auxiliary curves at their non-invariant points of intersection equals one (in Proposition 3.20, page 62). Subsection 3.4 (page 71) contains the statement and proof of the degree formula, in Theorem 3.22 (page 73).

Notation and Terminology

For ease of reference, we introduce and collect here the main notation and terminology that will be used throughout this paper.

- As usual, \mathbb{C} and \mathbb{R} correspond to the fields of complex and real numbers, respectively. The n -dimensional affine space is the set \mathbb{C}^3 , and the associated projective space will be denoted by \mathbb{P}^3 .
- We will use (y_1, y_2, y_3) for the affine coordinates in \mathbb{C}^3 , and $(y_0 : y_1 : y_2 : y_3)$ for the projective coordinates in \mathbb{P}^3 , as well as the abbreviations:

$$\bar{y} = (y_1, y_2, y_3), \quad \bar{y}_h = (y_0 : y_1 : y_2 : y_3).$$

- In order to distinguish offset surfaces from their generating surfaces, we will also use $\bar{x} = (x_1, x_2, x_3)$, $\bar{x}_h = (x_0 : x_1 : x_2 : x_3)$ to refer to the affine and projective coordinates of a point in the offset, and \bar{y}, \bar{y}_h as above for the original surface.
- A point in \mathbb{C}^3 will be denoted by

$$\bar{y}^o = (y_1^o, y_2^o, y_3^o)$$

and, correspondingly, a point in \mathbb{P}^3 will be denoted by

$$\bar{y}_h^o = (y_0^o : y_1^o : y_2^o : y_3^o)$$

Throughout this work, we will frequently use this o superscript to indicate a particular value of a variable.

- The Zariski closure of a set $A \subset \mathbb{C}^n$ will be denoted by A^* . The projective closure of an algebraic set A will be denoted by \bar{A} .
- Let A be an algebraic set. We denote by $\text{Sing}_a(A)$ the affine singular locus of A , and by $\text{Sing}(A)$ the projective singular locus of A ; i.e. the singular locus of \bar{A} .
- If I is a polynomial ideal, $\mathbf{V}(I)$ denotes the affine algebraic set defined by I ; that is,

$$\mathbf{V}(I) = \{\bar{x}^o \in \mathbb{C}^n / \forall f \in I, f(\bar{x}^o) = 0\}$$

- When we homogenize a polynomial $g \in \mathbb{C}[y_1, y_2, y_3]$, we will use capital letters, as in $G(y_0, y_1, y_2, y_3)$, to denote the homogenization of g w.r.t. y_0 . Also, by abuse of notation, we will write $g(\bar{y})$, $G(\bar{y}_h)$, $\mathbb{C}[\bar{y}]$, $\mathbb{C}[\bar{y}_h]$.
- The partial derivatives w.r.t. y_i of $g(y_1, y_2, y_3)$ and of its homogenization $G(y_0, y_1, y_2, y_3)$ will be denoted g_i and G_i respectively, for $i = 0, \dots, 3$. The symbol ∇g (resp. ∇G) denotes the gradient vector of partial derivatives, i.e.:

$$\nabla g(\bar{y}) = (g_1, g_2, g_3)(\bar{y}), \quad (\text{resp. } \nabla G(\bar{y}_h) = (G_1, \dots, G_3)(\bar{y}_h))$$

- The symbol Σ denotes a **rational algebraic surface** defined over \mathbb{C} by the irreducible polynomial $f(\bar{y}) \in \mathbb{C}[\bar{y}]$.
- We assume that we are given a **non-necessarily proper rational parametrization** of Σ :

$$P(\bar{t}) = \left(\frac{P_1(\bar{t})}{P_0(\bar{t})}, \frac{P_2(\bar{t})}{P_0(\bar{t})}, \frac{P_3(\bar{t})}{P_0(\bar{t})} \right).$$

Here $\bar{t} = (t_1, t_2)$, and $P_0, \dots, P_3 \in \mathbb{C}[\bar{t}]$ with $\gcd(P_0, \dots, P_3) = 1$.

- The **projectivization** P_h of P is obtained by homogenizing the components of P w.r.t. a new variable t_0 , multiplying both the numerators and denominators if necessary by a suitable power of t_0 . It will be denoted by

$$P_h(\bar{t}_h) = \left(\frac{X(\bar{t}_h)}{W(\bar{t}_h)}, \frac{Y(\bar{t}_h)}{W(\bar{t}_h)}, \frac{Z(\bar{t}_h)}{W(\bar{t}_h)} \right)$$

where $\bar{t}_h = (t_0 : t_1 : t_2)$, and $X, Y, Z, W \in \mathbb{C}[\bar{t}_h]$ are homogeneous polynomials of the same degree d_P , for which $\gcd(X, Y, Z, W) = 1$ holds.

- The (classical) offset at distance $d^o \in \mathbb{C}^\times$ to Σ is denoted by $\mathcal{O}_{d^o}(\Sigma)$ and the generic (classical) offset to Σ by $\mathcal{O}_d(\Sigma)$ (see Definition 1.13 in page 16). In this work, the variable d always represents the distance values.
- We denote by $g \in \mathbb{C}[d, \bar{x}]$ the generic offset equation for Σ (see Definition 1.16 in page 18).
- δ is the total degree of g w.r.t. \bar{x} ; i.e. $\delta = \deg_{\bar{x}}(g)$.
- Given $\phi(\bar{y}), \psi(\bar{y}) \in \mathbb{C}[\bar{y}]$ we denote by $\text{Res}_{y_i}(\phi, \psi)$ the univariate resultant of ϕ and ψ w.r.t. y_i , for $i = 0, \dots, n$. And if A is a subset of the set of variables $\{y_0, \dots, y_n\}$, we denote by $\text{PP}_A(\phi)$ (resp. $\text{Con}_A(\phi)$) the primitive part (resp. the content) of the polynomial ϕ w.r.t. A .
- $\mathcal{L}_{\bar{k}}$ denotes a generic line through the origin, whose direction is determined by the values of a variable $\bar{k} = (k_1, k_2, k_3)$. More precisely, for a particular value of \bar{k} , denoted by \bar{k}^o , the parametric equations of $\mathcal{L}_{\bar{k}}$ are

$$\ell_i(\bar{k}, l, \bar{x}) : x_i - k_i l = 0, \text{ for } i = 1, 2, 3,$$

where l is the parameter.

- We will keep the convention of always using the letter Δ to indicate a finite subset of values of the variable d . Accordingly, the letter Θ denotes a Zariski closed set of values of \bar{k} . A Zariski open subset of $\mathbb{C} \times \mathbb{C}^3$, formed by pairs of values of (d, \bar{k}) , will be denoted by Ω . In some proofs, an open set Ω will be constructed in several steps. In these cases we will use a superscript to indicate the step in the construction. Thus, $\Omega_1^0, \Omega_1^1, \Omega_1^2$, etc. are open sets, defined in successive steps in the construction of Ω_1 .

- A similar convention will be used for systems of equations and their solutions. A system of equations will be denoted by \mathfrak{S} , with sub and superscripts to distinguish between systems, and the set of solutions of the system will be denoted by Ψ , with the same choice of sub and superscripts.
- We will also need to consider local parametrizations of algebraic varieties. To distinguish local from (global) rational parametrizations, we will use calligraphic typeface for local parametrizations. Thus, a local parametrization will be denoted by

$$\mathcal{P}(\bar{t}) = (\mathcal{P}_1(\bar{t}), \dots, \mathcal{P}_n(\bar{t}))$$

- Let V be an n -dimensional vector space over \mathbb{C} . Two vectors $\bar{v} = (v_1, \dots, v_n)$ and $\bar{w} = (w_1, \dots, w_n)$ are said to be **parallel** if and only if

$$v_i w_j - v_j w_i = 0, \text{ for } i, j = 1, \dots, n$$

In this case we write $\bar{v} \parallel \bar{w}$.

- Given two vectors $(a_1, b_1, c_1), (a_2, b_2, c_2) \in \mathbb{C}^3$, their **cross product** is defined as

$$(a_1, b_1, c_1) \wedge (a_2, b_2, c_2) = (b_1 c_2 - b_2 c_1, a_2 c_1 - a_1 c_2, b_1 c_2 - b_2 c_1)$$

- We consider in V the symmetric bilinear form defined by:

$$\Xi(\bar{v}, \bar{w}) = \sum_{i=1}^n v_i w_i$$

which induces in V a metric vector space (see [30], [42]) with light cone of isotropy given by:

$$L_{\Xi} = \{(x_1, \dots, x_n) \in V / x_1^2 + \dots + x_n^2 = 0\}$$

Note that, when $\mathbb{C} = \mathbb{C}$, this is not the usual unitary space \mathbb{C}^n . On the other hand, when we consider the field \mathbb{R} , it is the usual Euclidean metric space, thus it preserves the usefulness of our results for applications. In this work, the **norm** $\|\bar{v}\|$ of a vector $\bar{v} \in V$ denotes a square root of $\bar{v} \cdot \bar{v}$, that is

$$\|\bar{v}\|^2 = \Xi(\bar{v}, \bar{v}) = \sum_{i=1}^n v_i^2$$

Moreover, a vector $\bar{v} \in V$ is **isotropic** if $\bar{v} \in L_{\Xi}$ (equivalently if $\|\bar{v}\| = 0$). Note that for a non-isotropic vectors there are precisely two choices of norm, which differ only by multiplication by -1 .

- We denote

$$\text{nor}_{(i,j)}(\bar{x}, \bar{y}) = f_i(\bar{y})(x_j - y_j) - f_j(\bar{y})(x_i - y_i).$$

Recall that $f(\bar{y})$ is the defining polynomial of the irreducible hypersurface Σ , and that $f_i(\bar{y})$ denotes its partial derivative w.r.t. y_i .

- The affine normal-hodograph of f is the polynomial:

$$h(\bar{y}) = f_1^2(\bar{y}) + f_2^2(\bar{y}) + f_3^2(\bar{y})$$

Following our convention of notation, H is the homogenization of h w.r.t. y_0 ; that is:

$$H(\bar{y}_h) = F_1^2(\bar{y}_h) + F_2^2(\bar{y}_h) + F_3^2(\bar{y}_h)$$

H is called the **projective normal-hodograph** of Σ . Moreover, a point $\bar{y}_h^o \in \bar{\Sigma}$ (resp. $\bar{y}^o \in \Sigma$) is called **normal-isotropic** if $H(\bar{y}_h^o) = 0$ (resp. $h(\bar{y}^o) = 0$).

- We denote by Σ_o the set of non normal-isotropic affine points of Σ ; that is:

$$\Sigma_o = \{\bar{y}^o \in \Sigma / h(\bar{y}^o) \neq 0\}$$

In the rest of this work we will assume that the Zariski-open subset Σ_o is non-empty. In [37], Proposition 2, it is proved that this is equivalent to H not being a multiple of F . We denote by $\text{Iso}(\Sigma)$ the closed set of affine normal-isotropic points of Σ . Note that $\text{Sing}_a(\Sigma) \subset \text{Iso}(\Sigma)$.

- If K is an irreducible component of an algebraic set A , and $K \subset \text{Iso}(A)$ we will say that K is normal-isotropic.

1 The Generic Offset

- In Subsection 1.1, we first review the concept of classical offset and the related properties that will be used in the sequel. In order to do this, we follow the incidence diagram formalism in [7] and [38] (see Diagram 2 in page 11; the reader may also see our previous papers [34] and [35])¹. Besides, the relation of this notion with Elimination Theory is established. We also collect several fundamental properties of the classical offset construction that we will refer to in the sequel. Next, the notions of generic offset and generic offset polynomial are introduced; see Definition 1.16 (page 18). The fundamental specialization property of the generic offset polynomial is established in Theorem 1.19 (page 19).

¹The referred works deal with general hypersurfaces. We consider here the special case of three dimensional surfaces.

- In Subsection 1.2, we recall some basic notions on parametric algebraic surfaces, and some technical lemmas about them. We also introduce the notion of associated normal vector, and we review some of its properties. Besides, we construct a parametric analogous of the Generic Offset System (see System 9).
- Subsection 1.3 (page 27) contains some technical results about the use of univariate resultants to study the problem of the intersection of plane algebraic curves. The classical setting for the computation of the intersection points of two plane curves by means of resultants is well known (see for instance [9], [43] and [40]). This requires in general a linear change of coordinates. However, in this work, we need to analyze the behavior of the resultant when some of the standard requirements are not satisfied. This is the content of Lemma 1.32 (page 28). Similarly, we also need to analyze the case when more than two curves are involved, by using generalized resultants. This is done in Lemma 1.33 (page 28).

1.1 Formal definition and basic properties of the generic offset

We begin by recalling the formal definition of classical offset, that can be found in [7] and [37]. With the notation introduced above, let $d^o \in \mathbb{C}^\times$ be a fixed value, and let $\Psi_{d^o}(\Sigma) \subset \mathbb{C}^7$ be the set of solutions (in the variables (\bar{x}, \bar{y}, u)) of the following polynomial system:

$$\left. \begin{array}{ll} \text{nor}_{(i,j)}(\bar{x}, \bar{y}) : & f(\bar{y}) = 0 \\ \text{(for } i, j = 1, \dots, 3; i < j) & f_i(\bar{y})(x_j - y_j) - f_j(\bar{y})(x_i - y_i) = 0 \\ b_{d^o}(\bar{x}, \bar{y}) : & (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 - (d^o)^2 = 0 \\ w(\bar{y}, u) : & u \cdot (\|\nabla f(\bar{y})\|^2) - 1 = 0 \end{array} \right\} \equiv \mathfrak{S}_1(d^o) \quad (1)$$

Let us consider the following:

$$\begin{array}{ccc} & \text{Offset Incidence Diagram} & \\ & \Psi_{d^o}(\Sigma) \subset \mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C} & \\ \swarrow \pi_1 & & \searrow \pi_2 \\ \mathcal{A}_{d^o}(\Sigma) \subset \mathbb{C}^3 & & \Sigma \subset \mathbb{C}^3 \end{array} \quad (2)$$

where

$$\begin{cases} \pi_1 : \mathbb{C}^7 \mapsto \mathbb{C}^3 \\ \pi_1(\bar{x}, \bar{y}, u) = \bar{x} \end{cases} \quad \text{and} \quad \begin{cases} \pi_2 : \mathbb{C}^7 \mapsto \mathbb{C}^3 \\ \pi_2(\bar{x}, \bar{y}, u) = \bar{y} \end{cases}$$

and $\mathcal{A}_{d^o}(\Sigma) = \pi_1(\Psi_{d^o}(\Sigma))$.

Definition 1.1. The (classical) offset to Σ at distance d° is the algebraic set $\mathcal{A}_{d^\circ}(\Sigma)^*$ (recall that the asterisk indicates Zariski closure). It will be denoted by $\mathcal{O}_{d^\circ}(\Sigma)$.

Remark 1.2.

1. If there is a solution of the system 1 of the form $\bar{p}^\circ = (\bar{x}^\circ, \bar{y}^\circ, u^\circ)$, then we say that the point $\bar{y}^\circ \in \Sigma$ and the point $\bar{x}^\circ \in \mathcal{O}_{d^\circ}(\Sigma)$ are associated points.
2. Let $I(d^\circ) \subset \mathbb{C}[\bar{x}, \bar{y}, u]$ be the ideal generated by the polynomials in $\mathfrak{S}_1(d^\circ)$; that is:

$$I(d^\circ) = \langle f(\bar{y}), b_{d^\circ}(\bar{x}, \bar{y}), \text{nor}_{(1,2)}(\bar{x}, \bar{y}), \text{nor}_{(1,3)}(\bar{x}, \bar{y}), \text{nor}_{(2,3)}(\bar{x}, \bar{y}), w(\bar{y}, u) \rangle.$$

This means that

$$\Psi_{d^\circ}(\Sigma) = \mathbf{V}(I(d^\circ))$$

is the affine algebraic set defined by $I(d^\circ)$, and

$$\mathcal{O}_{d^\circ}(\Sigma) = \mathbf{V}(\tilde{I}(d^\circ))$$

where $\tilde{I}(d^\circ) = I(d^\circ) \cap \mathbb{C}[\bar{x}]$ is the (\bar{y}, u) -elimination ideal of $I(d^\circ)$ (see [12], Closure Theorem, p. 122). In particular, this means that the offset can be computed by elimination techniques, such as Gröbner bases, resultants, characteristic sets, etc.

Next, we will refer to some properties of the classical offset construction, that we collect here for the reader's convenience. We start with a very important geometric property regarding the normal vector of the classical offset construction.

Proposition 1.3 (Fundamental Property of the Classical Offsets). Let $\bar{y}^\circ \in \mathcal{V}_o$, and let $\bar{x}^\circ \in \mathcal{O}_{d^\circ}(\mathcal{V})$ be a point associated to \bar{y}° . Then the normal line to \mathcal{V} at \bar{y}° is also normal to $\mathcal{O}_{d^\circ}(\mathcal{V})$ at \bar{x}° .

Proof. See [37], Theorem 16. □

In order to prove that we can avoid degenerated situations, we will sometimes need information about the dimension of certain sets of points. The basic tools for doing this will be the incidence diagrams, analogous to 2 (page 11) and well known results about the dimension of the fiber of a regular map. For ease of reference, we include here a statement of one such result, in a form that meets our needs. The proof can be found in [16]

Lemma 1.4. Let A be an affine algebraic set, and let $f : A \mapsto \mathbb{C}^p$ be a regular map. Let us denote $B = f(A)^*$. For $\bar{a}^\circ \in A$, let $\mu(\bar{a}^\circ) = \dim(f^{-1}(f(a^\circ)))$. Then, if $A_o \subset A$ is any irreducible component, $B_o = f(A_o)$ its image, and μ° is the minimum value of $\mu(\bar{a}^\circ)$ for $\bar{a}^\circ \in A_o$, we have

$$\dim(A_o) = \dim(B_o) + \mu^\circ$$

In particular, if there exists $a^\circ \in A_o$ for which $\dim(f^{-1}(f(a^\circ))) = 0$, then $\dim(A_o) = \dim(B_o)$.

We next analyze the number and dimension of the irreducible components of the offset.

Proposition 1.5. $\mathcal{O}_{d^o}(\mathcal{V})$ has at most two irreducible components.

Proof. See [37], Theorem 1. □

Proposition 1.6. The irreducible components of $\Psi_{d^o}(\mathcal{V})$ have the same dimension as \mathcal{V} .

Proof. In [37], Lemma 1, this is proved using local parametrizations. However, since, as we have seen above, π_2 is a $2 : 1$ map, this can also be considered a straightforward application of the preceding Lemma 1.4. □

Remark 1.7. This implies immediately that $\mathcal{O}_{d^o}(\mathcal{V})$ has at most two irreducible components, whose dimension is less or equal than $\dim(\mathcal{V})$.

To present the next two results, we recall some of the terminology introduced in [37]:

Definition 1.8.

1. The offset $\mathcal{O}_{d^o}(\mathcal{V})$ is called **degenerated** if at least one of its components is not a hypersurface.
2. A component $\mathcal{M} \subset \mathcal{O}_{d^o}(\mathcal{V})$ is said to be a **simple component** if there exists a non-empty Zariski dense subset $\mathcal{M}_1 \subset \mathcal{M}$ such that every point of \mathcal{M}_1 is associated to exactly one point of \mathcal{V} . Otherwise, \mathcal{M} is called a **special component** of the offset. Furthermore, we say that $\mathcal{O}_{d^o}(\mathcal{V})$ is **simple** if all its components are simple, and **special** if it has at least a special component (in this case, it has precisely one special component, see Proposition 1.10 below).

Note that if the offset is degenerated, and taking into account Lemma 1.4, the map π_1 must have a non-zero dimensional fiber for some point in $\Psi_{d^o}(\mathcal{V})$.

The following two results tell us that degeneration and special components are very infrequent phenomena.

Proposition 1.9. There is a finite set $\Delta_0 \subset \mathbb{K}$ such that if $d^o \notin \Delta_0$, then $\mathcal{O}_{d^o}(\mathcal{V})$ is not degenerated.

Proof. See [37], Theorem 2. □

Proposition 1.10.

1. Let \mathcal{M} be an irreducible and non-degenerated component of $\mathcal{O}_{d^o}(\mathcal{V})$. Then \mathcal{M} is special if and only if $\mathcal{O}_{d^o}(\mathcal{M}) = \mathcal{V}$.
2. $\mathcal{O}_{d^o}(\mathcal{V})$ has at least a simple component.
3. If $\mathcal{O}_{d^o}(\mathcal{V})$ is irreducible, then it is simple.
4. There is a finite set $\Delta_1 \subset \mathbb{C}$ such that, if $d^o \notin \Delta_1$ then $\mathcal{O}_{d^o}(\mathcal{V})$ is simple, and the irreducible components of $\mathcal{O}_{d^o}(\mathcal{V})$ are not contained in $\text{Iso}(\mathcal{V})$.

Proof. See [37], Theorems 7, 8 and Corollary 6. □

The next result shows that –as expected, being a metric construction– the offset construction is invariant under rigid motions of the affine space.

Proposition 1.11. *Let \mathcal{T} be a rigid motion of the affine space \mathbb{C}^3 . Then*

$$\mathcal{T}(\mathcal{O}_{d^o}(\mathcal{V})) = \mathcal{O}_{d^o}(\mathcal{T}(\mathcal{V}))$$

Proof. See [37], Lemma 2.5 in Chapter 2. □

Formal definition of the generic offset

The concept of generic offset to an algebraic surface was formally introduced in our previous paper [35]. The motivation for this concept is the following: As the distance value d^o varies, different offset varieties are obtained. The idea is to have a global expression of the offset for all (or almost all) distance values. This motivates the concept of **generic polynomial of the offset** to Σ . This is a polynomial, depending on the distance variable d , such that for every (or almost every, see the examples below) non-zero value d^o , the polynomial specializes to the defining polynomial of the offset at that particular distance. Let us see a couple of examples that give some insight into the situation.

Example 1.12.

- (a) *Using this informal definition of generic offset polynomial, and using Gröbner basis techniques, one can see that if \mathcal{C} is the parabola of equation $y_2 - y_1^2 = 0$, the generic polynomial of its offset is:*

$$\begin{aligned} g(d, x_1, x_2) = & -48 d^2 x_1^4 - 32 d^2 x_1^2 x_2^2 + 48 d^4 x_1^2 + 16 x_1^6 + 16 x_2^2 x_1^4 + 16 d^4 x_2^2 - \\ & 16 d^6 - 40 x_2 x_1^4 - 32 x_1^2 x_2^3 + 8 d^2 x_2 x_1^2 - 32 d^2 x_2^3 + 32 d^4 x_2 + x_1^4 + 32 x_1^2 x_2^2 + \\ & 16 x_2^4 - 20 d^2 x_1^2 - 8 d^2 x_2^2 - 8 d^4 - 2 x_2 x_1^2 - 8 x_2^3 + 8 x_2 d^2 + x_2^2 - d^2. \end{aligned}$$

In addition, and using again Gröbner basis techniques, one may check that for every distance the generic offset polynomial specializes properly (see Example 1.21 in page 21 below, for a detailed description of this example and the preceding claims).

- (b) On the other hand, the generic offset polynomial of the circle of equation $y_1^2 + y_2^2 - 1 = 0$ factors as the product of two circles of radius $1 + d$ and $1 - d$; that is:

$$g(d, x_1, x_2) = (x_1^2 + x_2^2 - (1 + d)^2) (x_1^2 + x_2^2 - (1 - d)^2).$$

Now, observe that for $d^o = 1$, this generic polynomial gives

$$g(1, x_1, x_2) = (x_1^2 + x_2^2 - 2^2) (x_1^2 + x_2^2) = (x_1^2 + x_2^2 - 2^2) (x_1 + ix_2) (x_1 - ix_2)$$

which describes the union of a circle of radius 2, and two complex lines. This is not a correct representation of the offset at distance 1 to \mathcal{C} , which consists of the union of the circle of radius 2 and a point (the origin). In fact, using Gröbner basis techniques, one has that the elimination ideal $\tilde{I}(1)$ (see Remark 1.2(2), page 12) is:

$$\tilde{I}(1) = \langle x_2(x_1^2 + x_2^2 - 4), x_1(x_1^2 + x_2^2 - 4) \rangle.$$

Thus, in this example we see that the generic offset polynomial does not specialize properly for $d^o = 1$. Nevertheless, for every other value of d^o the specialization is correct.

After these examples, we describe the formal definition of generic offset and generic offset polynomial, adapted to the case of surfaces in three dimensional space. We start by considering the following system of equations:

$$\left. \begin{array}{l} \text{nor}_{(i,j)}(\bar{x}, \bar{y}) : \\ \text{(for } i, j = 1, \dots, 3; i < j) \\ b(d, \bar{x}, \bar{y}) : \\ w(\bar{y}, u) : \end{array} \quad \begin{array}{l} f(\bar{y}) = 0 \\ f_i(\bar{y})(x_j - y_j) - f_j(\bar{y})(x_i - y_i) = 0 \\ (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 - d^2 = 0 \\ u \cdot (\|\nabla f(\bar{y})\|^2) - 1 = 0 \end{array} \right\} \equiv \mathfrak{S}_1(d) \quad (3)$$

The above system will be called the **Generic Offset System**. In this system, we consider d as a variable, so that $b \in \mathbb{C}[d, \bar{x}, \bar{y}]$. A solution of this system is thus a point of the form $(d^o, \bar{x}^o, \bar{y}^o, u^o) \in \mathbb{C}^8$.

Let $\Psi(\Sigma) \subset \mathbb{C} \times \mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}$ be the set of solutions of $\mathfrak{S}_1(d)$, and consider the following:

Generic Offset Incidence Diagram

$$\begin{array}{ccc}
 & \Psi(\Sigma) \subset \mathbb{C} \times \mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C} & \\
 \swarrow \pi_1 & & \searrow \pi_2 \\
 \mathcal{A}(\Sigma) \subset \mathbb{C}^4 & & \mathbb{C} \times \Sigma \subset \mathbb{C}^4
 \end{array} \tag{4}$$

where

$$\begin{cases} \pi_1 : \mathbb{C}^8 \mapsto \mathbb{C}^4 \\ \pi_1(d, \bar{x}, \bar{y}, u) = (d, \bar{x}) \end{cases} \quad \text{and} \quad \begin{cases} \pi_2 : \mathbb{C}^8 \mapsto \mathbb{C}^4 \\ \pi_2(d, \bar{x}, \bar{y}, u) = (d, \bar{y}) \end{cases}$$

and $\mathcal{A}(\Sigma) = \pi_1(\Psi(\Sigma))$.

Then one has the following definition (recall that the asterisk denotes the Zariski closure of a set):

Definition 1.13. *The generic offset to Σ is*

$$\mathcal{O}_d(\Sigma) = \mathcal{A}(\Sigma)^* = \pi_1(\Psi(\Sigma))^* \subset \mathbb{C}^4$$

Remark 1.14.

1. *Let*

$$I(d) = \langle f(\bar{y}), b(d, \bar{x}, \bar{y}), \text{nor}_{(1,2)}(\bar{x}, \bar{y}), \text{nor}_{(1,3)}(\bar{x}, \bar{y}), \text{nor}_{(2,3)}(\bar{x}, \bar{y}), w(\bar{y}, u) \rangle$$

be the ideal in $\mathbb{C}[d, \bar{x}, \bar{y}, u]$ generated by the polynomials in System 3. Note that the above definition implies that

$$\mathcal{O}_d(\Sigma) = \mathbf{V}(\tilde{I}(d))$$

where $\tilde{I}(d) = I(d) \cap \mathbb{C}[d, \bar{x}]$ is the (\bar{y}, u) -elimination ideal of $I(d)$.

2. *The Closure Theorem from Elimination Theory (see e.g. Theorem 3 in page 122 of [12]) implies that the dimension of the set*

$$\mathcal{O}_d(\Sigma) \setminus \pi_1(\Psi_1(\Sigma))$$

is smaller than the dimension of $\mathcal{O}_d(\Sigma)$. This is the set of points of the generic offset associated with singular or normal-isotropic points of Σ .

Basic properties of the generic offset

In the following Proposition we will see that the properties of the offset at a fixed distance, regarding its dimension and number of components (see Lemma 1, Theorem 1 and Theorem 2 in [37]), are reflected in the generic offset. In particular, this Proposition shows that the generic offset is a hypersurface, and thus guarantees the existence of the generic polynomial (see below, Definition 1.16).

Proposition 1.15.

1. $\mathcal{O}_d(\Sigma)$ has at most two components.
2. Each component of $\mathcal{O}_d(\Sigma)$ is a hypersurface in \mathbb{C}^4 .

Proof. (Adapted from Lemma 1, Theorem 1 and Theorem 2 in [37]). We begin by showing that if K is a component of $\Psi_1(\Sigma)$, then $\dim(K) = 3$. Thus

$$\dim(\Psi_1(\Sigma)) = 3 \tag{5}$$

Let $\psi^o = (d^o, \bar{x}^o, \bar{y}^o, u^o) \in K$. Then, $\bar{y}^o \in \Sigma$ is a regular point of Σ . Let $\mathcal{P}(\bar{t})$, with $\bar{t} = (t_1, \dots, t_{n-1})$, be a local parametrization of Σ at \bar{y}^o , with $\mathcal{P}(\bar{t}^o) = \bar{y}^o$. Then, it holds that one of the local parametrizations defined by:

$$\mathcal{P}^\pm(d, \bar{t}) = \left(d, \mathcal{P}(\bar{t}) \pm d \frac{\nabla f(\mathcal{P}(\bar{t}))}{\|\nabla f(\mathcal{P}(\bar{t}))\|}, \mathcal{P}(\bar{t}), \frac{1}{\|\nabla f(\mathcal{P}(\bar{t}))\|^2} \right)$$

parametrizes $\Psi_1(\Sigma)$ locally at ψ^o (we choose sign so that $\mathcal{P}^\pm(\bar{t}^o) = \psi^o$). Since $(d, \mathcal{P}(\bar{t}))$ parametrizes $\mathbb{C} \times \Sigma$, we get that (d, \bar{t}) are algebraically independent, and so $\dim(K) = 3$.

Now we can prove the first statement of the proposition. Since the number of components of $\mathcal{O}_d(\Sigma)$ is at most the number of components of $\Psi_1(\Sigma)$, one just only has to prove that $\Psi_1(\Sigma)$ has at most two components. Let us suppose that Γ_1, Γ_2 y Γ_3 are three different components of $\Psi_1(\Sigma)$ and let $Z = \pi_2(\Gamma_1) \cap \pi_2(\Gamma_2) \cap \pi_2(\Gamma_3)$, where π_2 is the projection of the incidence diagram 4. Then, it holds that $\dim(Z) = 3$. Observe that if $\dim(Z) < 3$ then $\dim(\Sigma \setminus Z) = \dim(\bigcup_{i=1}^3 (\Sigma \setminus \pi_2(\Gamma_i))) = 3$. Hence, at least one of the sets $\Sigma \setminus \pi_2(\Gamma_i)$ is of dimension 3, which is impossible since $\pi_2(\Gamma_i)$ are constructible sets of dimension 3. On the other hand, it holds that $\dim(Z \cap \pi_2(\Gamma_i \cap \Gamma_j)) < 3$ for $i < j$. Then

$$Z \setminus \bigcup_{i \neq j} (Z \cap \pi_2(\Gamma_i \cap \Gamma_j)) \neq \emptyset.$$

Now, take $\bar{p} = (d^o, \bar{y}^o) \in Z \setminus \bigcup_{i \neq j} (Z \cap \pi_2(\Gamma_i \cap \Gamma_j))$, then $\pi_2^{-1}(\bar{p}) = \{\bar{q}_1, \bar{q}_2, \bar{q}_3\}$ where $\bar{q}_i \neq \bar{q}_j$ for $i < j$, which is impossible since the mapping π_2 is $(2 : 1)$ on $\pi_2(\Psi_1(\Sigma))$.

Finally we can prove statement 2 in the proposition. We analyze the dimension of the tangent space to a component of the generic offset. Let $(d^o, \bar{y}^o) \in \pi_2(\Psi_1(\Sigma))$, such that

the two points $(d^o, \bar{x}_1^o), (d^o, \bar{x}_2^o) \in \mathcal{O}_d(\Sigma)$ generated by (d^o, \bar{y}^o) satisfy that the dimension of their tangent spaces is the dimension of the corresponding component of $\mathcal{O}_d(\Sigma)$. Let $u^o = \frac{1}{\|\nabla f(\bar{y}^o)\|^2}$, and let $\mathcal{P}(\bar{t})$ be a local parametrization of Σ at \bar{x}^o . Then, it holds that :

$$\tilde{\mathcal{P}}^+(d, \bar{t}) = \left(d, \mathcal{P}(\bar{t}) + d \frac{\nabla f(\mathcal{P}(\bar{t}))}{\|\nabla f(\mathcal{P}(\bar{t}))\|} \right) \quad \text{and} \quad \tilde{\mathcal{P}}^-(d, \bar{t}) = \left(d, \mathcal{P}(\bar{t}) - d \frac{\nabla f(\mathcal{P}(\bar{t}))}{\|\nabla f(\mathcal{P}(\bar{t}))\|} \right)$$

parametrize locally $\mathcal{O}_d(\Sigma)$ at (d^o, \bar{x}_1^o) , and (d^o, \bar{x}_2^o) . In this situation, let Q^\pm be as above, and consider the following map:

$$\begin{array}{ccccccc} \psi^+ : & \mathbb{C}^3 & \longrightarrow & \Psi_1(\Sigma) & \xrightarrow{\varphi^+} & \mathcal{A}(\Sigma) & \xhookrightarrow{i} \mathbb{C}^4 \\ & (d, \bar{t}) & \longrightarrow & \mathcal{P}^+(d, \bar{t}) & \longrightarrow & \tilde{\mathcal{P}}^+(d, \bar{t}) & \longrightarrow \tilde{\mathcal{P}}^+(d, \bar{t}). \end{array}$$

Similarly, we define ψ^- and φ^- . Now consider the following homomorphism, defined by the differential $d\psi^+$ (similarly for $d\psi^-$), between the tangent space to $\Psi_1(\Sigma)_1$ at $(d^o, \bar{x}_1^o, \bar{y}^o, u^o)$ and the tangent space $\mathcal{T}_{(d^o, \bar{x}_1^o)}$ to $\mathcal{A}(\Sigma)_1$ at (d^o, \bar{x}_1^o) , where $\Psi_1(\Sigma)_1$ and $\mathcal{A}(\Sigma)_1$ denote the component of $\Psi_1(\Sigma)$ and $\mathcal{A}(\Sigma)$ containing the points $(d^o, \bar{x}_1^o, \bar{y}^o, u^o)$ and (d^o, \bar{x}_1^o) , respectively. Then one has that

$$\dim(\mathcal{A}(\Sigma)_1) \geq \dim(\mathcal{T}_{(d^o, \bar{x}_1^o)}) \geq \dim(\text{Im}(d\varphi^+)) = \text{rank}(\mathcal{J}_{\varphi^+}),$$

where \mathcal{J}_{φ^+} denotes the jacobian matrix of φ^+ . Furthermore, by Equation 5 at the beginning of this proof, one has that

$$3 = \dim(\Psi_1(\Sigma)_1) \geq \dim(\mathcal{A}(\Sigma)_1) \geq \text{rank}(\mathcal{J}_{\varphi^+}).$$

On the other hand, if we take any point of the form $(0, \bar{t}^o) \in \mathbb{C}^n$, that is, with $d^o = 0$, we must get $\text{rank}(\mathcal{J}_{\varphi^+}) = 3$ at that point; otherwise, one would conclude that the rank of the jacobian of $\mathcal{P}(\bar{t})$ is smaller than 2, which is impossible since Σ is a surface. \square

As a first consequence of this Proposition, $\mathcal{O}_d(\Sigma)$ is defined by a polynomial $g(d, \bar{x}) \in \mathbb{C}[d, \bar{x}]$ (see [41], p.69, Theorem 3). Thus, we arrive at the following definition:

Definition 1.16. *The generic offset polynomial is the defining polynomial of the hypersurface $\mathcal{O}_d(\Sigma)$. In the sequel, we denote by $g(d, \bar{x})$ the generic offset equation.*

The first property of the generic offset polynomial that we study regards its factorization:

Lemma 1.17. *The generic offset polynomial is primitive w.r.t. \bar{x}*

Proof. Suppose, on the contrary, that $g(d, \bar{x})$ has a non-constant factor in $\mathbb{C}[d]$. That is

$$g(d, \bar{x}) = A(d)\tilde{g}(d, \bar{x})$$

Let $d^o \neq 0$ be any root of $A(d)$. Then the hypersurface \mathcal{Z} in $\mathbb{C} \times \mathbb{C}^3$ defined by $d = d^o$ is contained in $\mathcal{O}_d(\Sigma)$. Taking Remark 1.14(2) (page 16) into account, one has that there is an open non-empty subset of \mathcal{Z} contained in $\pi_1(\Psi_1(\Sigma))$. This in turn implies that there is an open subset $\tilde{\mathcal{Z}}$ of \mathbb{C}^3 such that if $\bar{x}^o \in \tilde{\mathcal{Z}}$, then $\bar{x}^o \in \mathcal{O}_{d^o}(\Sigma)$ (the classical offset at distance d^o). This is a contradiction, since we know that $\mathcal{O}_{d^o}(\Sigma)$ has dimension less or equal to 2. Thus, we are left with the case when $A(d)$ is a power of d . The argument must be different in this case, since the classical offset is only defined for $d^o \in \mathbb{C}^\times$. However, the reasoning is similar: we conclude that there is an open non-empty subset $\tilde{\mathcal{Z}}_0$ of \mathbb{C}^3 such that if $\bar{x}^o \in \tilde{\mathcal{Z}}_0$, then the system (“classical offset system for distance 0”)

$$\left. \begin{aligned} f(\bar{y}) &= 0 \\ f_i(\bar{y})(x_j^o - y_j) - f_j(\bar{y})(x_i^o - y_i) &= 0 \\ (\text{for } i, j = 1, \dots, 3; i < j) \\ (x_1^o - y_1)^2 + (x_2^o - y_2)^2 + (x_3^o - y_3)^2 &= 0 \\ u \cdot (\|\nabla f(\bar{y})\|^2) - 1 &= 0 \end{aligned} \right\} \equiv \mathfrak{S}_1(d)$$

has solutions. Now, if (\bar{y}^o, u^o) is a solution of this system then

1. $\nabla f(\bar{y}^o)$ is not isotropic,
2. $\bar{y}^o - \bar{x}^o$ is isotropic,
3. and $\nabla f(\bar{y}^o)$ is parallel to $\bar{y}^o - \bar{x}^o$,

Thus one has that $\bar{y}^o - \bar{x}^o = 0$. But since \bar{x}^o runs through an open subset of \mathbb{C}^3 , this contradicts the fact that Σ is a surface. \square

Remark 1.18.

1. Observe that the polynomial g may be reducible (recall the example of the circle) but by construction it is always square-free. Moreover, by Proposition 1.15 (page 17) and Lemma 1.17, g is either irreducible or factors into two irreducible factors not depending only on d .
2. We will also call $g(d, \bar{x}) = 0$ the generic offset equation of Σ .

The following theorem gives the fundamental property of the generic offset.

Theorem 1.19. *For all but finitely many exceptions, the generic offset polynomial specializes properly. That is, there exists a finite (possibly empty) set $\Delta_2 \subset \mathbb{C}$ such that if $d^o \notin \Delta_2$, then*

$$g(d^o, \bar{x}) = 0$$

is the equation of $\mathcal{O}_{d^o}(\Sigma)$.

Proof. Let $G(d)$ be a reduced Gröbner basis of $I(d)$ w.r.t. an elimination ordering that eliminates (\bar{y}, u) . Then, up to multiplication by a non-zero constant, $G(d) \cap \mathbb{C}[d, \bar{x}]$ is a Gröbner basis of $\tilde{I}(d)$. Proposition 1.15 above shows that $\tilde{G}(d) = G(d) \cap \mathbb{C}[d, \bar{x}] = \langle \nu(d)g(d, \bar{x}) \rangle$, where $\nu(d)$ is a non-zero polynomial, depending only on d (see the Remark preceding this proof). But then (see [12], exercise 7, page 283) there is a finite (possibly empty) set $\Delta_2^1 \subset \mathbb{C}$ such that for $d^o \notin \Delta_2^1$, $G(d)$ specializes well to a Gröbner basis of $I(d^o)$ (defined in Remark 1.2, page 12). It follows that, since $\tilde{I}(d^o) = I(d^o) \cap \mathbb{C}[\bar{x}]$, then $\tilde{G}(d^o) = \{\nu(d^o)g(d^o, \bar{x})\}$ is a Gröbner basis of $\tilde{I}(d^o)$. In particular, if Δ_2^2 is the finite set of zeros of $\nu(d)$, then for $d^o \notin \Delta_2 = \Delta_2^1 \cup \Delta_2^2$, and $d^o \neq 0$, one has that $g(d^o, \bar{x})$ is the equation for $\mathcal{O}_{d^o}(\Sigma)$. \square

For future reference, we collect in the following corollary all the information about the –finite– set of *bad* distances that appear in the offsetting construction.

Corollary 1.20. *There is a finite set $\Delta \subset \mathbb{C}^\times$ such that for $d^o \notin \Delta$, the following hold:*

- (1) *(non degeneracy): $\mathcal{O}_{d^o}(\Sigma)$ is not degenerated.*
- (2) *(simplicity): $\mathcal{O}_{d^o}(\Sigma)$ is simple.*
- (3) *(good specialization): if $g(d, \bar{x}) = 0$ is the generic offset polynomial, $g(d^o, \bar{x}) = 0$ is the equation of $\mathcal{O}_{d^o}(\Sigma)$.*
- (4) *(degree invariance):*

$$\deg_{\bar{x}}(\mathcal{O}_d(\Sigma)) = \deg_{\bar{x}}(\mathcal{O}_{d^o}(\Sigma)), \quad \deg_{x_i}(\mathcal{O}_d(\Sigma)) = \deg_{x_i}(\mathcal{O}_{d^o}(\Sigma)) \text{ for } i = 1, \dots, 3.$$

Proof. Take $\Delta^1 = \Delta_0 \cup \Delta_1 \cup \Delta_2$, with Δ_0 as in Proposition 1.9, Δ_1 as in Proposition 1.10(4) and Δ_2 as in Theorem 1.19 above. Furthermore, let $p(d)\bar{x}^\mu$ be a term of $g(d, \bar{x})$ of maximal degree w.r.t. \bar{x} . That is, $\bar{\mu} = (\mu_1, \mu_2, \mu_3) \in \mathbb{N}^3$, with $\sum \mu_i = \deg_{\bar{x}}(g)$, where $p(d) \in \mathbb{C}[d]$ is a non-zero polynomial. Then take:

$$\Delta^{\bar{x}} = \Delta \cup \{d^o \in \mathbb{C} \mid p(d^o) = 0\},$$

and similarly, for $i = 1, 2, 3$ construct $\Delta^{\bar{x}_i}$, by considering a term of $g(d, \bar{x})$ of maximal degree w.r.t x_i . Finally, taking

$$\Delta = \Delta^1 \cup \Delta^{\bar{x}} \cup \Delta^{\bar{x}_1} \cup \Delta^{\bar{x}_2} \cup \Delta^{\bar{x}_3},$$

our claim holds. \square

Let us see a first example of a generic offset polynomial.

Example 1.21. For the parabola \mathcal{C} with defining polynomial $f(y_1, y_2) = y_2 - y_1^2$, the generic offset system turns into:

$$\left. \begin{array}{l} \text{nor}_{(1,2)}(\bar{x}, \bar{y}) : \\ b(d, \bar{x}, \bar{y}) : \\ w(\bar{y}, u) : \end{array} \right\} \begin{array}{l} f(\bar{y}) = y_2 - y_1^2 \\ -2y_1(x_2 - y_2) - (x_1 - y_1) = 0 \\ (x_1 - y_1)^2 + (x_2 - y_2)^2 - d^2 = 0 \\ u \cdot (\|4y_1^2 + 1\|^2) - 1 = 0 \end{array}$$

Computing a Gröbner elimination basis of $I(d) = \langle f, \text{nor}_{(1,2)}, b, w \rangle$, we obtain (with the notation in the proof of Theorem 1.19):

$$G(d) = \{g(d, \bar{x}), \chi_1(d, \bar{x}), \dots, \chi_8(d, \bar{x})\}$$

where:

$$g(d, \bar{x}) = 16x_1^6 + 16x_1^4x_2^2 - 40x_1^4x_2 - 32x_1^2x_2^3 + (-48d^2 + 1)x_1^4 + (-32d^2 + 32)x_1^2x_2^2 + 16x_2^4 + (8d^2 - 2)x_1^2x_2 + (-32d^2 - 8)x_2^3 + (48d^4 - 20d^2)x_1^2 + (16d^4 - 8d^2 + 1)x_2^2 + (32d^4 + 8d^2)x_2 - 16d^6 - 8d^4 - d^2$$

and

$$\chi_1(d, \bar{x}) = 12d^2ux_1^2 + 16d^2ux_2^2 - 4d^2ux_2 + (-12d^4 + d^2)u - 4x_1^4 + 8x_1^2x_2 + 8d^2x_1^2 - 4x_2^2 + 4d^2x_2 - 4d^4 + 3d^2$$

$$\chi_2(d, \bar{x}) = 64d^2ux_2^3 - 48d^2ux_1^2 - 16d^2ux_2^2 + 28d^2ux_2 + (-60d^4 - 3d^2)u - 64x_1^4x_2 + 128x_1^2x_2^2 + 128d^2x_1^2x_2 - 64x_2^3 + 36d^2x_1^2 + 112d^2x_2^2 + (-64d^4 + 36d^2)x_2 - 36d^4 + 3d^2$$

$$\chi_3(d, \bar{x}) = 12y_2 - 16ux_1^2 - 16ux_2^2 + 8ux_2 + (16d^2 - 1)u - 8x_2 + 1$$

$$\chi_4(d, \bar{x}) = 12y_1x_2 + (-12d^2 - 3)y_1 - 8y_2x_1x_2 - 14y_2x_1 + 8x_1^3 + 8x_1x_2^2 - 6x_1x_2 + (-8d^2 + 3)x_1$$

$$\chi_5(d, \bar{x}) = 3y_1x_1 + 2y_2x_2 - y_2 - 2x_1^2 - 2x_2^2 + 2d^2$$

$$\chi_6(d, \bar{x}) = 12d^2u^2 - 4ux_1^2 - 16ux_2^2 - 4ux_2 + (4d^2 - 1)u + 4x_2 + 1$$

$$\chi_7(d, \bar{x}) = 12d^2y_1u + 8y_2ux_1x_2 + 14y_2ux_1 - 3y_1 - 8ux_1^3 - 8ux_1x_2^2 + 6ux_1x_2 + (8d^2 + 3)ux_1$$

$$\chi_8(d, \bar{x}) = y_1^2 + y_2^2 - 2y_1x_1 - 2y_2x_2 + x_1^2 + x_2^2 - d^2.$$

In particular,

$$G(d) \cap \mathbb{C}[d, \bar{x}] = \langle g(d, \bar{x}) \rangle.$$

And so $g(d, \bar{x})$ is the generic offset polynomial for the parabola \mathcal{C} .

This Gröbner basis has been computed considering the generators of $I(d)$ as polynomials in $\mathbb{C}(d)[\bar{x}, \bar{y}, u]$. This means that we have relationships of the form:

$$g(d, \bar{x}) = a_1(d)f(\bar{x}) + a_2(d)\text{nor}_{(1,2)}(\bar{x}, \bar{y}) + a_3(d)b(d, \bar{x}, \bar{y}) + a_4(d)w(\bar{y}, u)$$

and for $i = 1, \dots, 8$:

$$\chi_i(d, \bar{x}) = b_{i1}(d)f(\bar{x}) + b_{i2}(d)\text{nor}_{(1,2)}(\bar{x}, \bar{y}) + b_{i3}(d)b(d, \bar{x}, \bar{y}) + b_{i4}(d)w(\bar{y}, u)$$

where $a_1, \dots, a_4, b_{11}, \dots, b_{84} \in \mathbb{C}(d)$. The result in Exercise 7, in page 283 of [12] indicates that the Gröbner basis specializes well for all values d^o such that none of

the denominators of the rational functions a_i and b_{ij} vanish at d^o . In this particular example, one may compute these rational functions and check that they are all constant. Therefore, specializing $g(d, \bar{x})$ provides the offset equation for every non-zero value of d . The computations in this example were obtained with the computer algebra system Singular (see [15]). We do not include here the details of the computations, because of obvious space limitations.

The following result, about the dependence on d of the generic offset polynomial, is an easy consequence of Theorem 1.19 above. In fact, Theorem 1.19 implies that there are infinitely many values d^o such that $g(d^o, \bar{x})$ is the polynomial of $\mathcal{O}_{d^o}(\Sigma)$ and, simultaneously, $g(-d^o, \bar{x})$ is the polynomial of $\mathcal{O}_{-d^o}(\Sigma)$. But, because of the symmetry in the construction, the offsets $\mathcal{O}_{d^o}(\Sigma)$ and $\mathcal{O}_{-d^o}(\Sigma)$ are exactly the same algebraic set. Thus, it follows that for infinitely many values of d^o it holds that up to multiplication by a non-zero constant:

$$g(d^o, \bar{x}) = g(-d^o, \bar{x}).$$

Hence, we have proved the following proposition:

Proposition 1.22. *The generic offset polynomial belongs to $\mathbb{C}[\bar{x}][d^2]$. That is, it only contains even powers of d .*

Now we can describe precisely the central problem of this work.

Remark 1.23. *The total degree problem for Σ consists of finding (efficient) formulae to compute the total degree of g in the variables \bar{x} . We denote this total degree by δ .*

1.2 Surface Parametrizations and Parametric Offset System

Since Σ is an algebraic surface over \mathbb{C} , all of its irreducible components have dimension 2 over \mathbb{C} . Besides, assuming that the surface Σ is **unirational (or parametric)** means that there exists a rational map $P : \mathbb{C}^2 \mapsto \Sigma$ such that the image of P is dense in Σ w.r.t. the Zariski topology. The map P is called an **(affine) parametrization** of Σ . If P is a birational map, then Σ is called a **rational surface**, and P is called a **proper parametrization** of Σ . In this paper we will not assume that P is proper (see Lemma 1.24 in page 23, and the observations preceding it). It is well known that a rational surface is always irreducible.

Thus, a parametrization P of Σ is given through a non-constant triplet of rational functions in two parameters. We will use $\bar{t} = (t_1, t_2)$ for the parameters of P and, as usual, $\bar{t}^o = (t_1^o, t_2^o)$ stands for a particular value in \mathbb{C}^2 of the pair of parameters. By a simple algebraic manipulation, we can assume that the three components of P have a

common denominator. Thus, we can write:

$$P(\bar{t}) = \left(\frac{P_1(\bar{t})}{P_0(\bar{t})}, \frac{P_2(\bar{t})}{P_0(\bar{t})}, \frac{P_3(\bar{t})}{P_0(\bar{t})} \right) \quad (6)$$

where $P_0, \dots, P_3 \in \mathbb{C}[\bar{t}]$ and $\gcd(P_0, \dots, P_3) = 1$. The number

$$d_P = \max_{i=0, \dots, 3} (\{\deg_{\bar{t}}(P_i)\})$$

is then called the **degree** of P .

Over the algebraically closed field \mathbb{C} , the notions of rational and parametric surface are equivalent (see the Castelnuovo Theorem [10]). Furthermore, there exists an algorithm by Schicho (see [36]) to obtain a proper parametrization of a rational surface given by its implicit equation. Thus, in principle, given a non-proper parametrization of a surface, it is possible (though computationally very expensive) to implicitize, and then apply Schicho's algorithm to obtain a proper parametrization. In addition, [24] shows how to properly reparametrize certain special families of rational surfaces. However, **in this paper we will not assume that P is proper**, and the degree formulas below take this fact into account.

The parametrization P has two **associated tangent vectors**, denoted by

$$\frac{\partial P(\bar{t})}{\partial t_1} \text{ and } \frac{\partial P(\bar{t})}{\partial t_2}. \quad (7)$$

That is:

$$\frac{\partial P}{\partial t_i} = \left(\frac{P_{1,i}P_0 - P_1P_{0,i}}{(P_0)^2}, \frac{P_{2,i}P_0 - P_2P_{0,i}}{(P_0)^2}, \frac{P_{3,i}P_0 - P_3P_{0,i}}{(P_0)^2} \right)$$

where $P_{j,i}$ denotes the partial derivative of P_j w.r.t. t_i , for $j = 0, \dots, 2$ and $i = 1, 2$.

The following Lemma states those properties of the surface parametrization P that we will need in the sequel.

Lemma 1.24. *There are non-empty Zariski open subsets $\Upsilon_1 \subset \mathbb{C}^2$ and $\Upsilon_2 \subset \Sigma$ such that:*

$$P : \Upsilon_1 \mapsto \Upsilon_2$$

is a surjective regular application of degree m . In particular, this means that P defines a $m : 1$ correspondence between Υ_1 and Υ_2 . Thus, given $\bar{y}^o \in \Upsilon_2$, there are precisely m different values $\bar{t}_1^o, \dots, \bar{t}_m^o$ of the parameter \bar{t} such that $P(\bar{t}_i^o) = \bar{y}^o$ for $i = 1, \dots, m$. Furthermore, if $\bar{t}^o \in \Upsilon_1$, the rank of the Jacobian matrix $\left(\frac{\partial P}{\partial \bar{t}} \right)$ evaluated at \bar{t}^o is two.

Proof. See e.g. [25]. □

Remark 1.25.

1. The number m is also called, as in the case of curves, the **tracing index** of P . See [40] for an algorithm to compute m . In the sequel, we will denote by m the tracing index of P .
2. As a consequence of this lemma, the part of the surface Σ not covered by the image of P is a proper closed subset (i.e. a finite collection of curves and points).

Starting with the parametrization P of Σ as in (6) above, we will construct a polynomial normal vector to Σ , that will be used in the statements of the degree formulas for rational surfaces. This particular choice of normal vector will be called in the sequel the **associated normal vector** of P , and it will be denoted by $\bar{n}(\bar{t})$.

To construct $\bar{n}(\bar{t})$, we first take the cross product of the associated tangent vectors introduced in 7, page 23. Let us denote:

$$V(\bar{t}) = \frac{\partial P(\bar{t})}{\partial t_1} \wedge \frac{\partial P(\bar{t})}{\partial t_2}$$

This vector $V(\bar{t})$ has the following form:

$$V(\bar{t}) = \left(\frac{A_1(\bar{t})}{A_0(\bar{t})}, \frac{A_2(\bar{t})}{A_0(\bar{t})}, \frac{A_3(\bar{t})}{A_0(\bar{t})} \right)$$

where $A_i \in \mathbb{C}[\bar{t}]$. Let $G(\bar{t}) = \gcd(A_1, A_2, A_3)$.

Definition 1.26. With the above notation, the **associated normal vector** $\bar{n} = (n_1, n_2, n_3)$ to P is the vector whose components are the polynomials:

$$n_i(\bar{t}) = \frac{A_i(\bar{t})}{G(\bar{t})} \text{ for } i = 1, 2, 3.$$

Remark 1.27.

1. Note that \bar{n} is a normal vector to Σ at $P(\bar{t})$, vanishing at most at a finite set of points in the \bar{t} plane. To see this observe that, because of their construction, n_1, n_2, n_3 have no common factors. Besides, at most one of the polynomials n_i is constant (otherwise the surface is a plane). Thus, the non constant components of \bar{n} define a system of at least two plane curves without common components.
2. In particular, there are some $\mu \in \mathbb{N}$ and $\beta(\bar{t}) \in \mathbb{C}[\bar{t}]$, with $\gcd(\beta, P_0) = 1$, such that

$$f_i(P(\bar{t})) = \frac{\beta(\bar{t})}{P_0(\bar{t})^\mu} n_i(\bar{t}) \text{ for } i = 1, 2, 3. \quad (8)$$

That is:

$$\nabla f(P(\bar{t})) = \frac{\beta(\bar{t})}{P_0(\bar{t})^\mu} \cdot \bar{n}(\bar{t})$$

3. Note that the polynomial $\beta(\bar{t})$ introduced above is not identically zero. Otherwise, one has $f_i(P(\bar{t})) = 0$ for $i = 1, 2, 3$, and this implies that $f(\bar{y})$ is a constant polynomial, which is a contradiction.

Definition 1.28. The polynomial $h \in \mathbb{C}[\bar{t}]$ defined as

$$h(\bar{t}) = n_1(\bar{t})^2 + n_2(\bar{t})^2 + n_3(\bar{t})^2$$

is called the **parametric (affine) normal-hodograph** of the parametrization P .

Remark 1.29. In the sequel, if we need to refer to the implicit normal-hodograph introduced in page 10, we will denote it by H_{imp} in the projective case, resp. h_{imp} in the affine case.

The following lemma will be used below to exclude from our discussion certain pathological cases, associated to some particular parameter values.

Lemma 1.30. The sets Υ_1 and Υ_2 in Lemma 1.24 (page 23) can be chosen so that if $\bar{t}^o \in \Upsilon_1$, then

$$P_0(\bar{t}^o)h(\bar{t}^o)\beta(\bar{t}^o) \neq 0.$$

In particular, $\bar{n}(\bar{t}^o) \neq 0$.

Proof. Note that P_0 , h and β are non-zero polynomials. Thus, the equation:

$$P_0(\bar{t})h(\bar{t})\beta(\bar{t}) = 0$$

defines an algebraic curve. Let us call it \mathcal{C} . Then it suffices to replace Υ_1 (resp. Υ_2) in Lemma 1.24 with $\Upsilon_1 \setminus \mathcal{C}$ (resp. $\Upsilon_2 \setminus P(\mathcal{C})$). \square

Parametric system for the generic offset

Let Σ and P be as above. In order to describe $\mathcal{O}_d(\Sigma)$ from a parametric point of view, we introduce the following system, to be called the **parametric system for the generic offset**:

$$\mathfrak{S}_1^P(d) \equiv \begin{cases} b^P(d, \bar{t}, \bar{x}) : (P_0x_1 - P_1)^2 + (P_0x_2 - P_2)^2 + (P_0x_3 - P_3)^2 - d^2P_0^2 = 0 \\ \text{nor}_{(1,2)}^P(\bar{t}, \bar{x}) : n_1 \cdot (P_0x_2 - P_2) - n_2 \cdot (P_0x_1 - P_1) = 0 \\ \text{nor}_{(1,3)}^P(\bar{t}, \bar{x}) : n_1 \cdot (P_0x_3 - P_3) - n_3 \cdot (P_0x_1 - P_1) = 0 \\ \text{nor}_{(2,3)}^P(\bar{t}, \bar{x}) : n_2 \cdot (P_0x_3 - P_3) - n_3 \cdot (P_0x_2 - P_2) = 0 \\ w^P(r, \bar{t}) : r \cdot P_0 \cdot h \cdot \beta - 1 = 0 \end{cases} \quad (9)$$

Our first result will show that this system provides an alternative description for the generic offset. To state this, we will introduce some additional notation. Let

$$\Psi^P \subset \mathbb{C} \times \mathbb{C} \times \mathbb{C}^2 \times \mathbb{C}^3$$

be the set of solutions, in the variables (d, r, \bar{t}, \bar{x}) , of the system $\mathfrak{S}_1^P(d)$. We also consider the projection maps

$$\begin{cases} \pi_1^P : \mathbb{C} \times \mathbb{C} \times \mathbb{C}^2 \times \mathbb{C}^3 \mapsto \mathbb{C} \times \mathbb{C}^3 \\ \pi_1^P(d, r, \bar{t}, \bar{x}) = (d, \bar{x}) \end{cases} \quad \text{and} \quad \begin{cases} \pi_2^P : \mathbb{C} \times \mathbb{C} \times \mathbb{C}^2 \times \mathbb{C}^3 \mapsto \mathbb{C} \times \mathbb{C}^2 \\ \pi_2^P(d, r, \bar{t}, \bar{x}) = (r, \bar{t}) \end{cases}$$

and we define $\mathcal{A}^P = \pi_1^P(\Psi^P)$. Recall that $(\mathcal{A}^P)^*$ denotes the Zariski closure of \mathcal{A}^P .

Proposition 1.31.

$$\mathcal{O}_d(\Sigma) = (\mathcal{A}^P)^*.$$

Proof. With the notation introduced in Definition 1.13, page 16, recall that

$$\mathcal{O}_d(\Sigma) = \mathcal{A}(\Sigma)^* = \pi_1(\Psi(\Sigma))^*.$$

Note that in this proof we use π_1, π_2 as in page 16, to be distinguished from π^P, π_2^P introduced above. Let Υ_1, Υ_2 be as in Lemma 1.30, page 25, and let us denote

$$\mathcal{B}_\Sigma^P = \pi_2^{-1}(\mathbb{C} \times \Upsilon_2).$$

\mathcal{B}_Σ^P is a non-empty dense subset of $\Psi(\Sigma)$, because $\mathbb{C} \times \Upsilon_2$ is dense in $\mathbb{C} \times \Sigma$. It follows that $\mathcal{O}_d(\Sigma) = \pi_1(\mathcal{B}_\Sigma^P)^*$. We will show that $\pi_1(\mathcal{B}_\Sigma^P) = \mathcal{A}^P$, thus completing the proof.

If $(d^o, \bar{x}^o) \in \pi_1(\mathcal{B}_\Sigma^P)$, there are \bar{y}^o, u^o and $\bar{t}^o \in \Upsilon_1$ such that $(d^o, \bar{x}^o, \bar{y}^o, u^o) \in \Psi(\Sigma)$, with $\bar{y}^o = P(\bar{t}^o)$. Since $u^o \neq 0$ and also $P_0(\bar{t}^o)h(\bar{t}^o)\beta(\bar{t}^o) \neq 0$, we can define:

$$r^o = \frac{u^o \beta(\bar{t}^o)}{P_0(\bar{t}^o)^{2\mu+1}}.$$

where μ is as in Equation 8, page 24. Then, substituting $P(\bar{t}^o)$ by \bar{y}^o in System 9, and using also Equation 8, one has that:

$$\begin{cases} b^P(d^o, \bar{t}^o, \bar{x}^o) = P_0(\bar{t}^o)^2 b(d^o, \bar{x}^o, \bar{y}^o) = 0 \\ \text{nor}_{(i,j)}^P(\bar{t}^o, \bar{x}^o) = \frac{P_0(\bar{t}^o)^{\mu+1}}{\beta(\bar{t}^o)} \text{nor}_{(i,j)}(\bar{x}^o, \bar{y}^o) = 0 \\ w^P(r^o, \bar{t}^o) = w(u^o, \bar{y}^o) = 0, \end{cases} \quad (10)$$

because $(d^o, \bar{x}^o, \bar{y}^o, u^o) \in \Psi(\Sigma)$. Therefore, one concludes that $(d^o, r^o, \bar{t}^o, \bar{x}^o) \in \Psi^P$, and so $(d^o, \bar{x}^o) \in \mathcal{A}^P$. This proves that $\pi_1(\mathcal{B}_\Sigma^P) \subset \mathcal{A}^P$.

Conversely, let $(d^o, \bar{x}^o) \in \mathcal{A}^P$. Then, there are \bar{t}^o, r^o such that $(d^o, r^o, \bar{t}^o, \bar{x}^o) \in \Psi^P$. Since $P_0(\bar{t}^o)h(\bar{t}^o)\beta(\bar{t}^o) \neq 0$,

$$\bar{y}^o = P(\bar{t}^o) \quad \text{and} \quad u^o = \frac{r^o P_0(\bar{t}^o)^{2\mu+1}}{\beta(\bar{t}^o)}$$

are well defined. The equations (10) still hold, and in this case, they imply that $(d^o, \bar{x}^o, \bar{y}^o, u^o) \in \Psi(\Sigma)$. Besides, $\pi_2(d^o, \bar{x}^o, \bar{y}^o, u^o) = (d^o, \bar{y}^o) \in \mathbb{C} \times \Upsilon_2$, and so $(d^o, \bar{x}^o) \in \pi_1(\mathcal{B}_\Sigma^P)$. This proves that $\mathcal{A}^P \subset \pi_1(\mathcal{B}_\Sigma^P)$, thus finishing the proof. \square

1.3 Intersection of Curves and Resultants

In the following sections we will show that we can translate the offset degree problem into a suitably constructed planar curves intersection problem. In this subsection we gather some results about the planar curves intersection problem to be used in the sequel.

It is well known that the intersection points of two plane curves, without common components, as well as their multiplicity of intersection, can be computed by means of resultants. For this, a suitable preparatory change of coordinates may be required (see for instance, [9], [43] and, for a modern treatment of the subject, [40]). In this work, for reasons that will turn out to be clear in subsequent sections, we need to analyze the behavior of the resultant factors, and their correspondence with multiplicities of intersection, when some of the standard requirements are not satisfied. Similarly, we also need to analyze the case when more than two curves are involved.

More precisely, we will use two technical lemmas. The first one, whose proof can be found in [33], shows that, under certain conditions, the multiplicity of intersection is reflected in the factors appearing in the resultant, even though the curves are not properly set. In particular, the requirement that no two intersection points lie on a line through the origin can be relaxed, obtaining in this case the total multiplicity of intersection along that line.

The second lemma, Lemma 1.33, is a generalization of Corollary 1 in [25]. It shows that generalized resultants can be used to study the intersection points of a finite family of curves. This lemma will be applied in Section 3 to the case of surfaces.

As we have said in the preceding paragraphs, the multiplicity of intersection of two projective plane curves can be read at the resultant of their defining polynomials. In fact, this is often used to define the multiplicity of intersection. More precisely (see [40]), let \mathcal{C}_1 and \mathcal{C}_2 be projective plane curves, without common components, such that $(1 : 0 : 0) \notin \mathcal{C}_1 \cup \mathcal{C}_2$, and $(1 : 0 : 0)$ does not belong to any line connecting two points in $\mathcal{C}_1 \cap \mathcal{C}_2$. Let $F(y_0, y_1, y_2)$, resp. $G(y_0, y_1, y_2)$, be the defining polynomials of \mathcal{C}_1 , resp.

\mathcal{C}_2 . Let $\bar{y}_h^o = (y_0^o : y_1^o : y_2^o) \in \mathcal{C}_1 \cap \mathcal{C}_2$, and let

$$R(y_1, y_2) = \text{Res}_{y_0}(F, G)$$

Then the multiplicity of intersection of \mathcal{C}_1 and \mathcal{C}_2 at \bar{y}_h^o , denoted by $\text{mult}_{\bar{y}_h^o}(\mathcal{C}_1, \mathcal{C}_2)$, equals the multiplicity of the corresponding factor $(y_2^o y_1 - y_1^o y_2)$ in $R(y_1, y_2)$. However, in the following Lemma we see how the multiplicity of intersection of two curves on a line through the origin can be read in the resultant, under certain circumstances, even though the curves are not properly set. This lemma can be seen as a generalization of Theorem 5.3, page 111 in [43].

Lemma 1.32. *Let \mathcal{C}_1 and \mathcal{C}_2 be two projective algebraic plane curves without common components, given by the homogeneous polynomials $F(y_0, y_1, y_2)$ and $G(y_0, y_1, y_2)$, respectively. Let p_1, \dots, p_k be the intersection points, different from $(1 : 0 : 0)$, of \mathcal{C}_1 and \mathcal{C}_2 lying on the line of equation $\beta y_1 - \alpha y_2 = 0$. Then the factor $(\beta y_1 - \alpha y_2)$ appears in $\text{Res}_{y_0}(F, G)$ with multiplicity equal to*

$$\sum_{i=1}^k \text{mult}_{p_i}(\mathcal{C}_1, \mathcal{C}_2)$$

Proof. See Lemma 19 in [33]. □

The following Lemma is a generalization of Corollary 1 in [25]. It shows that generalized resultants can be used to study the intersection points of a finite family of curves.

Lemma 1.33. *Let $\mathcal{C}_0, \dots, \mathcal{C}_m$ be the projective plane curves, defined by the homogeneous polynomials $F_0, \dots, F_m \in \mathbb{C}[\bar{t}_h]$, respectively. Let us suppose that the following hold:*

- (i) F_1, \dots, F_m have positive degree in t_0 .
- (ii) $\deg_{\bar{t}_h}(F_1) = \dots = \deg_{\bar{t}_h}(F_m)$.
- (iii) $\gcd(F_1, \dots, F_m) = 1$.

Let us denote:

$$F(\bar{c}, \bar{t}_h) = c_1 F_1(\bar{t}^h) + \dots + c_m F_m(\bar{t}^h)$$

and let

$$R(\bar{c}, \bar{t}) = \text{Res}_{t_0}(F_0(\bar{t}^h), F(\bar{c}, \bar{t}_h)),$$

(note that by (iii), $R(\bar{c}, \bar{t})$ is not identically zero). Finally, let $\text{lc}_{t_0}(F_0) \in \mathbb{C}[\bar{t}]$ and $\text{lc}_{t_0}(F) \in \mathbb{C}[\bar{c}, \bar{t}]$ denote, respectively, the leading coefficients w.r.t. t_0 of F_0 and F .

If $\bar{t}^o = (t_1^o, t_2^o) \in \mathbb{C}^2 \setminus \{0\}$ is such that $\text{Cont}_{\bar{c}}(R)(\bar{t}^o) = 0$ and

$$\text{lc}_{t_0}(F_0)(\bar{t}^o) \cdot \text{lc}_{t_0}(F)(\bar{c}, \bar{t}^o) \neq 0,$$

there exists t_0^o such that $\bar{t}_h^o = (t_0^o : t_1^o : t_2^o) \in \bigcap_{i=0}^m \mathcal{C}_i$.

Proof. First, observe that if $\deg_{t_0}(F_0) = 0$, then $\text{lc}_{t_0}(F_0) = F_0$ and $R(\bar{c}, \bar{t}) = F_0^{\deg_{\bar{t}_0}(F_1)}$. Thus, in this case the lemma holds trivially, since there is no $\bar{t}^o \in \mathbb{C}^2 \setminus \{\bar{0}\}$ satisfying the hypothesis of the lemma. Thus, w.l.o.g., in the rest of the proof, we assume that $\deg_{t_0}(F_0) > 0$.

Since $\text{lc}_{t_0}(F)(\bar{c}, \bar{t}^o) \neq 0$, there exists an open set $\Phi \subset \mathbb{C}^m$ such that if $\bar{c}^o = (c_1^o, \dots, c_m^o) \in \Phi$, the leading coefficient w.r.t. t_0 of $F(\bar{c}^o, \bar{t}^o, t_0) \in \mathbb{C}[t_0]$ is $\text{lc}_{t_0}(F)(\bar{c}^o, \bar{t}^o)$, and it is non-zero. Therefore, by the Extension Theorem, (see [12], page 159), there exists $\zeta(\bar{c}^o) \in \mathbb{C}$ (which, in principle, could depend on \bar{c}^o) such that

$$F_0(\zeta(\bar{c}^o), t_1^o, t_2^o) = F(\zeta(\bar{c}^o), t_1^o, t_2^o) = 0.$$

We claim that there is $t_0^o \in \mathbb{C}$ (not depending on \bar{c}^o), such that

$$F_0(t_0^o, t_1^o, t_2^o) = F_1(t_0^o, t_1^o, t_2^o) = \dots = F_m(t_0^o, t_1^o, t_2^o) = 0.$$

To see this note that, since $\text{lc}_{t_0}(F_0) \neq 0$, there is a non-empty finite set of solutions of the following equation in t_0 :

$$F_0(t_0, t_1^o, t_2^o) = 0.$$

Let ζ_1, \dots, ζ_p be the solutions. If

$$F_1(\zeta_j, t_1^o, t_2^o) = \dots = F_m(\zeta_j, t_1^o, t_2^o) = 0$$

holds for some $j = 1, \dots, p$, then it suffices to take $t_0^o = \zeta_j$. Let us suppose that this is not the case, and we will derive a contradiction. Then there exists an open set $\Phi_1 \subset \Phi$, such that if $\bar{c}^o \in \Phi_1$, then

$$F(\zeta_j, t_1^o, t_2^o) = c_1^o F_1(\zeta_j, t_1^o, t_2^o) + \dots + c_m^o F_m(\zeta_j, t_1^o, t_2^o) \neq 0$$

for every $j = 1, \dots, p$. This means that, for $\bar{c}^o \in \Phi_1$, there is no solution of:

$$\begin{cases} F_0(t_0, \bar{t}^o) = 0 \\ F(\bar{c}^o, t_0, \bar{t}^o) = 0 \end{cases}$$

Since the resultant specializes properly in Φ , this implies that: $R(\bar{c}^o, \bar{t}^o) \neq 0$. But, denoting

$$M(\bar{t}) = \text{Cont}_{\bar{c}}(R(\bar{c}, \bar{t})), \quad \text{and } N(\bar{c}, \bar{t}) = \text{PP}_{\bar{c}}(R(\bar{c}, \bar{t})),$$

we have

$$R(\bar{c}^o, \bar{t}^o) = M(\bar{t}^o)N(\bar{c}^o, \bar{t}^o) = 0$$

because, by hypothesis $M(\bar{t}^o) = 0$. This contradiction proves the result. \square

2 Offset-Line Intersection for Parametric Surfaces

As we said in the introduction to this paper, the parametric character of Σ results in a reduction of the dimension of the space in which we count the points in $\mathcal{O}_d(\Sigma) \cap \mathcal{L}_{\bar{k}}$. This is so because, instead of counting directly those points, we count the values of the \bar{t} parameters that generate them. In this section we will show how, with this approach, we are led to an intersection problem between projective plane curves, and we will analyze that problem. More precisely:

- Subsection 2.1 (page 30) is devoted to the analysis of the intersection between the generic offset and a pencil of lines through the origin. The results in this subsection (see Theorem 2.8, page 35) constitute the theoretical foundation of the degree formula to be derived in Section 3.
- In Subsection 2.2 we describe the auxiliary polynomials obtained by using elimination techniques in the Parametric Offset-Line System, and we introduce a new auxiliary system, see System 12. Also, we obtain some geometric properties of the solutions of this new system $\mathfrak{S}_3^P(d, \bar{k})$ in Proposition 2.11 (page 40) and the subsequent Lemma 2.13 (page 43). These results will be used in the sequel to elucidate the relation between the solution sets of Systems $\mathfrak{S}_2^P(d, \bar{k})$ and $\mathfrak{S}_3^P(d, \bar{k})$.
- In Subsection 2.3 (page 44) we define the corresponding notion of fake points and invariant points for the Affine Auxiliary System $\mathfrak{S}_3^P(d, \bar{k})$. The main result of this subsection is Proposition 2.18 (page 46), that shows the relation between these two notions.

2.1 Intersection with lines

As in the case of plane curves (see our paper [33]), we will address the degree problem for surfaces by counting the number of intersection points between $\mathcal{O}_d(\Sigma)$ and a generic line through the origin. More precisely, let us consider a family of lines through the origin, denoted by $\mathcal{L}_{\bar{k}}$, whose direction is determined by the values of the variable $\bar{k} = (k_1, k_2, k_3)$. The family $\mathcal{L}_{\bar{k}}$ is described by the following set of parametric equations:

$$\mathcal{L}_{\bar{k}} \equiv \begin{cases} \ell_1(\bar{k}, l, \bar{x}) : x_1 - k_1 l = 0 \\ \ell_2(\bar{k}, l, \bar{x}) : x_2 - k_2 l = 0 \\ \ell_3(\bar{k}, l, \bar{x}) : x_3 - k_3 l = 0 \end{cases}$$

A particular line of the family, corresponding to the value \bar{k}^o , will be denoted by $\mathcal{L}_{\bar{k}^o}$. We add the equations ℓ_1, ℓ_2, ℓ_3 of $\mathcal{L}_{\bar{k}}$ to the equations of the parametric system for the

generic offset (System 9 in page 25), and we arrive at the following system:

$$\mathfrak{S}_2^P(d, \bar{k}) \equiv \begin{cases} b^P(d, \bar{t}, \bar{x}) : (P_0x_1 - P_1)^2 + (P_0x_2 - P_2)^2 + (P_0x_3 - P_3)^2 - d^2P_0^2 = 0 \\ \text{nor}_{(1,2)}^P(\bar{t}, \bar{x}) : n_1 \cdot (P_0x_2 - P_2) - n_2 \cdot (P_0x_1 - P_1) = 0 \\ \text{nor}_{(1,3)}^P(\bar{t}, \bar{x}) : n_1 \cdot (P_0x_3 - P_3) - n_3 \cdot (P_0x_1 - P_1) = 0 \\ \text{nor}_{(2,3)}^P(\bar{t}, \bar{x}) : n_2 \cdot (P_0x_3 - P_3) - n_3 \cdot (P_0x_2 - P_2) = 0 \\ w^P(r, \bar{t}) : r \cdot P_0 \cdot \beta \cdot h - 1 = 0 \\ \ell_1(\bar{k}, l, \bar{x}) : x_1 - k_1 l = 0 \\ \ell_2(\bar{k}, l, \bar{x}) : x_2 - k_2 l = 0 \\ \ell_3(\bar{k}, l, \bar{x}) : x_3 - k_3 l = 0 \end{cases} \quad (11)$$

We will refer to this as the **Parametric Offset-Line System**. The next step is the study of the generic solutions of this system.

We need to exclude certain degenerate situations that arise for a set of values of (d, \bar{k}) . For example, a degenerated situation arises if the set of points of Σ where the normal line to Σ passes through the origin is *too big*. The next Lemma says that this can only happen if Σ is a sphere centered at the origin.

Lemma 2.1. *Let $\Sigma_\perp \subset \Sigma$ denote the set of regular points $\bar{y}^o \in \Sigma$ such that the normal line to Σ at \bar{y}^o is parallel to \bar{y}^o . If Σ is not a sphere centered at the origin, then Σ_\perp^* is a proper (possibly empty) closed subset of Σ .*

Proof. Let us assume that Σ_\perp is nonempty. Let, as usual, $f(\bar{y})$ be the irreducible polynomial defining Σ , and let $\tilde{\Sigma}$ be the algebraic set in \mathbb{C}^3 defined by:

$$\begin{cases} f(\bar{y}) = 0 \\ f_i(\bar{y})y_j - f_j(\bar{y})y_i = 0 \quad (\text{for } i, j = 1, \dots, 3; i < j). \end{cases}$$

Note that this set of equations implies $\bar{y}^o \parallel \nabla f(\bar{y}^o)$ for $\bar{y}^o \in \Sigma$. Then $\Sigma_\perp \subset \tilde{\Sigma} \subset \Sigma$. Therefore, it suffices to prove that $\tilde{\Sigma} \neq \Sigma$. Let us suppose that $\tilde{\Sigma} = \Sigma$. Let

$$K(\bar{y}) = \bar{y} \cdot \nabla f(\bar{y}) = \sum_{j=1}^3 y_j f_j(\bar{y}).$$

Then for every $\bar{y}^o \in \Sigma$, using that $f_i(\bar{y}^o)y_j^o = f_j(\bar{y}^o)y_i^o$ one has that

$$f_i(\bar{y}^o)K(\bar{y}^o) = \sum_{j=1}^3 f_i(\bar{y}^o)y_j^o f_j(\bar{y}^o) = y_i^o \sum_{j=1}^3 f_j(\bar{y}^o)^2 = y_i^o h(\bar{y}^o),$$

for $i = 1, 2, 3$. Now let $\bar{t} = (t_1, t_2)$ and let $\mathcal{Q}(\bar{t}) = (Q_1, Q_2, Q_3)(\bar{t})$ be a local parametrization of Σ . Substituting \mathcal{Q} in the above relation:

$$f_i(\mathcal{Q}(\bar{t}))K(\mathcal{Q}(\bar{t})) = Q_i(\bar{t})h(P(\bar{t}))$$

that is, $K(\mathcal{Q}(\bar{t}))\nabla f(\mathcal{Q}(\bar{t})) = h(\mathcal{Q}(\bar{t}))\mathcal{Q}(\bar{t})$. Using Prop. 2 in [37], we know that $h(\mathcal{Q}(\bar{t})) \neq 0$, and so $K(\mathcal{Q}(\bar{t})) \neq 0$. Thus:

$$\frac{h(\mathcal{Q}(\bar{t}))}{K(\mathcal{Q}(\bar{t}))}Q_i(\bar{t}) = f_i(\mathcal{Q}(\bar{t})).$$

On the other hand, since $f(\mathcal{Q}(\bar{t})) = 0$, deriving w.r.t. t_j , ($j = 1, 2$) one has:

$$\sum_{i=1}^3 f_i(\mathcal{Q}(\bar{t}))\frac{\partial Q_i(\bar{t})}{\partial t_j} = \frac{h(\mathcal{Q}(\bar{t}))}{K(\mathcal{Q}(\bar{t}))} \sum_{i=1}^3 Q_i(\bar{t})\frac{\partial Q_i(\bar{t})}{\partial t_j} = 0$$

From this, one concludes that

$$\frac{\partial}{\partial t_j} \left(\sum_{i=1}^3 Q_i^2(\bar{t}) \right) = 2 \sum_{i=1}^3 Q_i(\bar{t})\frac{\partial Q_i(\bar{t})}{\partial t_j} = 0$$

for $j = 1, 2$. This means that $\sum_{i=1}^3 Q_i^2(\bar{t}) = c$ for some constant $c \in \mathbb{C}$. Since Σ is assumed not to be normal-isotropic, one has $c \neq 0$, and since the parametrization converges locally, we conclude that Σ equals a sphere centered at the origin. \square

Remark 2.2. Note that, if in Lemma 2.1 we consider those regular points \bar{y}^o of Σ such that the normal line to Σ at \bar{y}^o is parallel to the vector $\bar{y}^o - \bar{a}$ for a fixed $\bar{a} \in \mathbb{C}^3$, then Σ_{\perp}^* is a proper (possibly empty) closed subset of Σ , unless Σ is a sphere centered at \bar{a} .

A closer analysis of the proof of Lemma 2.1 shows that in fact we have also proved the following:

Corollary 2.3. If \mathcal{W} is any irreducible component of Σ_{\perp}^* with $\dim(\mathcal{W}) > 0$ then \mathcal{W} is contained in a sphere centered at the origin. That is, there exists $d^o \in \mathbb{C}^\times$ such that if $\bar{y}^o \in \mathcal{W}$, then

$$(y_1^o)^2 + (y_2^o)^2 + (y_3^o)^2 = (d^o)^2.$$

Since Σ_{\perp}^* has at most finitely many irreducible components, it follows that there is a finite set of distances $\{d_1^\perp, \dots, d_p^\perp\}$ such that Σ_{\perp}^* is contained in the union the spheres centered at the origin and with radius d_i^\perp for $i = 1, \dots, p$.

We will use the notation $\Upsilon(\Sigma_{\perp}) = \{d_1^\perp, \dots, d_p^\perp\}$, and we will say that $\Upsilon(\Sigma_{\perp})$ is the set of critical distances of Σ .

The following lemma is the basic tool to avoid the remaining degenerated situations in the analysis of the offset-line intersection: for a given proper closed subset $\mathfrak{F} \subset \Sigma$, it shows

1. how to avoid the set of values (d^o, \bar{k}^o) such that $\mathcal{L}_{\bar{k}^o} \setminus \{\bar{0}\}$ meets Σ in a point $\bar{y}^o \in \mathfrak{F}$,
2. and how to avoid the set of values (d^o, \bar{k}^o) such that $\mathcal{L}_{\bar{k}^o} \setminus \{\bar{0}\}$ meets $\mathcal{O}_{d^o}(\Sigma)$ in a point \bar{x}^o associated to $\bar{y}^o \in \mathfrak{F}$.

In the proof of the Lemma we will use the polynomials f, h, b and $\text{nor}_{(i,j)}$ (for $i, j = 1, \dots, 3; i < j$), introduced with System $\mathfrak{S}_1(d)$ in page 15. For the convenience of the reader we repeat that system here:

$$\left. \begin{array}{ll} \text{nor}_{(i,j)}(\bar{x}, \bar{y}) : & f_i(\bar{y})(x_j - y_j) - f_j(\bar{y})(x_i - y_i) = 0 \\ \text{(for } i, j = 1, \dots, 3; i < j) & \\ b(d, \bar{x}, \bar{y}) : & (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 - d^2 = 0 \\ w(\bar{y}, u) : & u \cdot (\|\nabla f(\bar{y})\|^2) - 1 = 0 \end{array} \right\} \equiv \mathfrak{S}_1(d).$$

and that $h(\bar{t}) = n_1(\bar{t})^2 + n_2(\bar{t})^2 + n_3(\bar{t})^2$, while $h_{\text{imp}}(\bar{t}) = \|\nabla f(\bar{y})\|^2$.

Lemma 2.4. *Let $\mathfrak{F} \subsetneq \Sigma$ be closed. There exists an open $\Omega_{\mathfrak{F}} \subset \mathbb{C}^4$, such that if $(d^o, \bar{k}^o) \in \Omega_{\mathfrak{F}}$, the following hold:*

- (1) $\mathcal{L}_{\bar{k}^o} \cap (\mathfrak{F} \setminus \{\bar{0}\}) = \emptyset$.
- (2) If $\bar{x}^o \in (\mathcal{L}_{\bar{k}^o} \cap \mathcal{O}_{d^o}(\Sigma)) \setminus \{\bar{0}\}$, there is no solution $(d^o, \bar{x}^o, \bar{y}^o, u^o)$ of System $\mathfrak{S}_1(d)$ (System 3 in page 15) with $\bar{y}^o \in \mathfrak{F}$.

Proof. If \mathfrak{F} is empty, the result is trivial. Thus, let us assume that $\mathfrak{F} \neq \emptyset$, and let the defining polynomials of \mathfrak{F} be $\{\phi_1(\bar{y}), \dots, \phi_p(\bar{y})\} \subset \mathbb{C}[\bar{y}]$. We will show that one may take $\Omega_{\mathfrak{F}} = \Omega_{\mathfrak{F}}^1 \cap \Omega_{\mathfrak{F}}^2$, where $\Omega_{\mathfrak{F}}^1, \Omega_{\mathfrak{F}}^2$ are two open sets constructed as follows:

- (a) Let us consider the following ideal in $\mathbb{C}[\bar{k}, \rho, v, \bar{y}]$:

$$\mathcal{I} = \langle f(\bar{y}), \phi_1(\bar{y}), \dots, \phi_p(\bar{y}), \bar{y} - \rho \cdot \bar{k}, v \cdot \rho - 1 \rangle,$$

and the projection maps defined in its solution set $\mathbf{V}(\mathcal{I})$ as follows:

$$\pi_{(1,1)}(\bar{k}, \rho, v, \bar{y}) = \bar{y}, \quad \pi_{(1,2)}(\bar{k}, \rho, v, \bar{y}) = \bar{k}$$

We show first that $\pi_{(1,1)}(\mathbf{V}(\mathcal{I})) = \mathfrak{F}$. The inclusion $\pi_{(1,1)}(\mathbf{V}(\mathcal{I})) \subset \mathfrak{F}$ is trivial; and if $\bar{y}^o \in \mathfrak{F}$, then since $\mathcal{F} \subset \Sigma$, $(\bar{y}^o, 1, 1, \bar{y}^o) \in \mathbf{V}(\mathcal{I})$ proves the reversed inclusion. Therefore, since $\mathcal{F} \subsetneq \Sigma$, $\dim(\pi_{(1,1)}(\mathbf{V}(\mathcal{I}))) = \dim(\mathfrak{F}) < 2$. Besides, for every $\bar{y}^o \in \pi_{(1,1)}(\mathbf{V}(\mathcal{I}))$ one has:

$$\pi_{(1,1)}^{-1}(\bar{y}^o) = \{(v^o \bar{y}^o, \frac{1}{v^o}, v^o, \bar{y}^o) \mid v^o \in \mathbb{C}^\times\}$$

from where one has that $\dim(\pi_{(1,1)}^{-1}(\bar{y}^o)) = 1$. Since the dimension of the fiber does not depend on \bar{y}^o , applying Lemma 1.4 (page 12), we obtain $\dim(\mathbf{V}(\mathcal{I})) < 3$. Thus, $\dim(\pi_{(1,2)}(\mathbf{V}(\mathcal{I}))) < 3$. It follows that $(\pi_{(1,2)}(\mathbf{V}(\mathcal{I})))^*$ is a proper closed subset of \mathbb{C}^3 . Let $\Theta^1 = \mathbb{C}^3 \setminus (\pi_{(1,2)}(\mathbf{V}(\mathcal{I})))^*$, and let $\Omega_{\mathfrak{F}}^1 = \mathbb{C} \times \Theta^1$.

(b) Let us consider the following ideal in $\mathbb{C}[d, \bar{k}, \rho, v, \bar{x}, \bar{y}]$:

$$\begin{aligned} \mathcal{J} = & \langle f(\bar{y}), b(d, \bar{x}, \bar{y}), \text{nor}_{(1,2)}(\bar{x}, \bar{y}), \text{nor}_{(1,3)}(\bar{x}, \bar{y}), \text{nor}_{(2,3)}(\bar{x}, \bar{y}), \\ & \bar{x} - \rho \cdot \bar{k}, v \cdot \rho \cdot d \cdot h_{\text{imp}}(\bar{y}) - 1, \phi_1(\bar{y}), \dots, \phi_p(\bar{y}) \rangle \end{aligned}$$

and the projection maps defined in its solution set $\mathbf{V}(\mathcal{J}) \subset \mathbb{C}^{12}$ as follows:

$$\pi_{(2,1)}(d, \bar{k}, \rho, v, \bar{x}, \bar{y}) = \bar{y}, \quad \pi_{(2,2)}(d, \bar{k}, \rho, v, \bar{x}, \bar{y}) = (d, \bar{k})$$

Then $\pi_{(2,1)}(\mathbf{V}(\mathcal{J})) \subset \mathfrak{F}$. Therefore $\dim(\pi_{(2,1)}(\mathbf{V}(\mathcal{J}))) \leq 1$. Let $\bar{y}^o \in \pi_{(2,1)}(\mathbf{V}(\mathcal{J}))$. Note that then $h_{\text{imp}}(\bar{y}^o) \neq 0$. We denote $\sigma^o = \sqrt{h_{\text{imp}}(\bar{y}^o)}$ (a particular choice of the square root); clearly $\sigma^o \neq 0$. Then, it holds that:

$$\pi_{(2,1)}^{-1}(\bar{y}^o) = \left\{ \left(d^o, \frac{1}{\rho^o} \left(\bar{y}^o \pm \frac{d^o}{\sigma^o} \nabla(\bar{y}^o) \right), \rho^o, \frac{1}{(\sigma^o)^2 \rho^o d^o}, \bar{y}^o \pm \frac{d^o}{\sigma^o} \nabla(\bar{y}^o), \bar{y}^o \right) \mid d^o, \rho^o \in \mathbb{C}^\times \right\}$$

Therefore $\dim(\pi_{(2,1)}^{-1}(\bar{y}^o)) = 2$. Applying Lemma 1.4 again, one has

$$\dim(\mathbf{V}(\mathcal{J})) = 2 + \dim(\pi_{(2,1)}(\mathbf{V}(\mathcal{J}))) \leq 3.$$

It follows that $\dim(\pi_{(2,2)}(\mathbf{V}(\mathcal{J}))) \leq 3$. Let us take $\Omega_{\mathfrak{F}}^2 = \mathbb{C}^4 \setminus \pi_{(2,2)}(\mathbf{V}(\mathcal{J}))^*$.

Let $\Omega_{\mathfrak{F}} = \Omega_{\mathfrak{F}}^1 \cap \Omega_{\mathfrak{F}}^2$ and let $(d^o, \bar{k}^o) \in \Omega_{\mathfrak{F}}$.

1. If $\bar{y}^o \in \mathcal{L}_{\bar{k}^o} \cap (\mathfrak{F} \setminus \{\bar{0}\})$, then there is some $\rho^o \in \mathbb{C}^\times$ such that $\bar{y}^o = \rho^o \bar{k}^o$. It follows that $(\bar{k}^o, \rho^o, \frac{1}{\rho^o}, \bar{y}^o) \in \mathbf{V}(\mathcal{I})$, and so $\bar{k}^o \in \pi_{(1,2)}(\mathbf{V}(\mathcal{I}))$, contradicting the construction of $\Omega_{\mathfrak{F}}^1$. This proves statement (1).
2. If $\bar{x}^o \in (\mathcal{L}_{\bar{k}^o} \cap \mathcal{O}_{d^o}(\Sigma)) \setminus \{\bar{0}\}$, and there is a solution $(d^o, \bar{x}^o, \bar{y}^o, u^o)$ of System $\mathfrak{G}_1(d)$ with $\bar{y}^o \in \mathfrak{F}$, then there is some $\rho^o \in \mathbb{C}^\times$ such that $\bar{x}^o = \rho^o \bar{k}^o$. It follows that $(d^o, \bar{k}^o, \rho^o, \frac{1}{\rho^o \cdot d^o \cdot h_{\text{imp}}(\bar{y}^o)}, \bar{x}^o, \bar{y}^o) \in \mathbf{V}(\mathcal{J})$. Therefore $(d^o, \bar{k}^o) \in \pi_{(2,2)}(\mathbf{V}(\mathcal{J}))$, contradicting the construction of $\Omega_{\mathfrak{F}}^2$. This proves statement (2).

□

Remark 2.5. Note that the origin may belong to \mathfrak{F} . In that case, Lemma 2.4(1) guarantees that the origin is the only point in $\mathcal{L}_{\bar{k}^o} \cap \mathcal{F}$. Correspondingly, part (2) of the lemma guarantees that the remaining points in $\mathcal{L}_{\bar{k}^o} \cap \mathcal{O}_{d^o}(\Sigma)$ cannot be extended to a solution $(d^o, \bar{x}^o, \bar{y}^o, u^o)$ of System $\mathfrak{G}_1(d)$ with $\bar{y}^o \in \mathfrak{F}$.

Our next goal is to prove a theorem (Theorem 2.8 below), that gives the theoretical foundation for our approach to the degree problem. Theorem 2.8 is the analogous of Theorem 5 in our paper [33]. That theorem is preceded by Lemma 4, that states that for a curve \mathcal{C} , $\bar{0} \in \mathcal{O}_{d^o}(\mathcal{C})$ for at most finitely many values $d^o \in \mathbb{C}$. However, we have not been able to prove a similar result for the case of surfaces: the main difficulty is that a surface can have infinitely many singular points. Even if we restrict ourselves to the case of parametric surfaces, we still have to take into account the possible existence of a singular curve contained in Σ , and not contained in the image of the parametrization. Besides, in the proof of the theorem we will use Lemma 2.1 (page 31), that does not apply when Σ is a sphere centered at the origin. This is the reason for the Assumptions that we announced in the Introduction of this paper, and that we state formally here. In the sequel, we assume that:

Assumptions 2.6.

- (1) There exists a finite subset Δ^1 of \mathbb{C} such that, for $d^o \notin \Delta^1$ the origin does not belong to $\mathcal{O}_{d^o}(\Sigma)$.
- (2) Σ is not a sphere centered at the origin.

Before stating the theorem we have to introduce some terminology.

Remark 2.7. For $(d^o, \bar{k}^o) \in \mathbb{C}^4$ we will denote by $\Psi_2^P(d^o, \bar{k}^o)$ the set of solutions of System $\mathfrak{S}_2^P(d^o, \bar{k}^o)$ in the variables (l, r, \bar{t}, \bar{x}) (see (11) in page 31).

Theorem 2.8. Let Σ satisfy the hypothesis in Remark 2.6. There exists a non-empty Zariski-open subset $\Omega_0 \subset \mathbb{C}^4$, such that if $(d^o, \bar{k}^o) \in \Omega_0$, then

- (a) if $\bar{y}^o \in \mathcal{L}_{\bar{k}^o} \cap (\Sigma \setminus \{\bar{0}\})$, then no normal vector to Σ at \bar{y}^o is parallel to \bar{y}^o .
- (b) $\Psi_2^P(d^o, \bar{k}^o)$ has precisely $m\delta$ elements (recall that m is the tracing index of P and δ the total degree of the generic offset). Besides, the set $\Psi_2^P(d^o, \bar{k}^o)$ can be partitioned as a disjoint union:

$$\Psi_2^P(d^o, \bar{k}^o) = \Psi_2^1(d^o, \bar{k}^o) \cup \dots \cup \Psi_2^\delta(d^o, \bar{k}^o),$$

such that:

- (b1) $\#\Psi_2^i(d^o, \bar{k}^o) = m$ for $i = 1, \dots, \delta$.
- (b2) The m elements of $\#\Psi_2^i(d^o, \bar{k}^o)$ have the same values of the variables (l, r, \bar{x}) , and differ only in the value of \bar{t} . Besides, for $(l^o, r^o, \bar{t}^o, \bar{x}^o) \in \Psi_2^i(d^o, \bar{k}^o)$, the point $P(\bar{t}^o) \in \Sigma$ does not depend on the choice of \bar{t}^o .

Let us denote by $(l_i^o, r_i^o, \bar{t}_{h,i}^o, \bar{x}_i^o)$ an element of $\Psi_2^i(d^o, \bar{k}^o)$. Then

(b3) The points $\bar{x}_1^o, \dots, \bar{x}_\delta^o$ are all different (and different from $\bar{0}$), and

$$\mathcal{L}_{\bar{k}^o} \cap \mathcal{O}_{d^o}(\Sigma) = \{\bar{x}_1^o, \dots, \bar{x}_\delta^o\}.$$

Furthermore, \bar{x}_i^o is non normal-isotropic in $\mathcal{O}_{d^o}(\Sigma)$, for $i = 1, \dots, \delta$.

(b4) The δ points

$$\bar{y}_1^o = P(\bar{t}_{h,1}^o), \dots, \bar{y}_\delta^o = P(\bar{t}_{h,\delta}^o)$$

are affine, distinct and non normal-isotropic points of Σ .

(c) $k_i^o \neq 0$ for $i = 1, 2, 3$.

Proof. Let $\Delta_0^1 = \{d^o \in \mathbb{C} \mid g(d^o, \bar{0}) \neq 0\}$. The assumption in Remark 2.6 (page 35) implies that Δ_0^1 is an open non-empty subset of \mathbb{C} . Let Δ be as in Corollary 1.20, (page 20), and let $\Omega_0^0 = (\Delta_0^1 \cap (\mathbb{C} \setminus \Delta)) \times (\mathbb{C}^3 \setminus (\{\bar{k}^o/k_i^o = 0 \text{ for some } i = 1, 2, 3\}))$. Next, let us consider $g(d, \bar{x})$ expressed as follows:

$$g(d, \bar{x}) = \sum_{i=0}^{\delta} g_i(d, \bar{x})$$

where g_i is a degree i form in \bar{x} . We consider:

$$\tilde{g}(d, \bar{k}, \rho) = g(d, \rho \bar{k}) = \sum_{i=0}^{\delta} g_i(d, \bar{k}) \rho^i.$$

This polynomial is not identically zero, is primitive w.r.t. \bar{x} (see Lemma 1.17, page 18), and it is squarefree; note that $g(d, \bar{x})$ is square-free by Remark 1.18 (page 19), and therefore \tilde{g} is square-free too. Thus, the discriminant

$$Q(d, \bar{k}) = \text{Dis}_\rho(\tilde{g}(d, \bar{k}, \rho))$$

is not identically zero either.

In this situation, let us take

$$\Omega_1^0 = \Omega_0^0 \setminus \{(d^o, \bar{k}^o) \in \mathbb{C}^4 / Q(d^o, \bar{k}^o) \cdot g_0(d^o, \bar{k}^o) \cdot g_\delta(d^o, \bar{k}^o) = 0\}.$$

For $(d^o, \bar{k}^o) \in \Omega_1^0$, $g(d^o, \rho \bar{k}_0)$ has δ different and non-zero roots; say, $\rho_1, \dots, \rho_\delta$. Therefore, $\mathcal{L}_{\bar{k}^o}$ intersects $\mathcal{O}_{d^o}(\Sigma)$ in δ different points:

$$\bar{x}_1^o = \rho_1 \bar{k}^o, \dots, \bar{x}_\delta^o = \rho_\delta \bar{k}^o,$$

and none of these points is the origin.

We will now construct an open subset $\Omega_2^0 \subset \Omega_1^0$ such that for $(d^o, \bar{k}^o) \in \Omega_2^0$, the points $\bar{x}_1^o, \dots, \bar{x}_\delta^o$ are non-normal isotropic points in $\mathcal{O}_{d^o}(\Sigma)$, and each one of them is associated

with a unique non-normal isotropic point of Σ . To do this, recall that $\text{Iso}(\Sigma)$ is the closed set of normal-isotropic points of Σ (see page 10), and let $\Omega_{\text{Iso}(\Sigma)}$ be the set obtained when applying Lemma 2.4 (page 33) to the closed subset $\mathfrak{F} = \text{Iso}(\Sigma)$. Note that if $(d^o, \bar{k}^o) \in \Omega_1^0 \cap \Omega_{\text{Iso}(\Sigma)}$, then the points $\bar{x}_1^o, \dots, \bar{x}_\delta^o$ are not associated with normal-isotropic points of Σ . Let us consider the polynomial

$$\Gamma(d, \bar{x}) = \left(\frac{\partial g}{\partial x_1}(d, \bar{x}) \right)^2 + \left(\frac{\partial g}{\partial x_2}(d, \bar{x}) \right)^2 + \left(\frac{\partial g}{\partial x_3}(d, \bar{x}) \right)^2.$$

This polynomial is not identically zero, because in that case for every $d^o \notin \Delta$ all the points in $\mathcal{O}_{d^o}(\Sigma)$ would be normal-isotropic, contradicting Proposition 1.10(3) (page 14). Let then $\tilde{\Gamma}(d, \bar{k}, r) = \Gamma(d, r\bar{k})$, and consider the resultant:

$$\Phi(d, \bar{k}) = \text{Res}_r(\tilde{g}(d, \bar{k}, r), \tilde{\Gamma}(d, \bar{k}, r))$$

If $\Phi(d, \bar{k}) \equiv 0$, then $\tilde{g}(d, \bar{k}, r)$ y $\tilde{\Gamma}(d, \bar{k}, r)$ have a common factor of positive degree in r . Let us show that this leads to a contradiction. Suppose that

$$\begin{cases} \tilde{g}(d, \bar{k}, r) = M(d, \bar{k}, r)G(d, \bar{k}, r), \\ \tilde{\Gamma}(d, \bar{k}, r) = M(d, \bar{k}, r)\Gamma^*(d, \bar{k}, r). \end{cases}$$

Then M depends on \bar{k} (because \tilde{g} cannot have a non constant factor in $\mathbb{C}[d, r]$). Take therefore $r^o \in \mathbb{C}^\times$ such that $M(d, \frac{\bar{k}}{r^o}, r^o)$ depends on \bar{k} . Then:

$$\begin{cases} g(d, \bar{x}) = g(d, r^o \frac{\bar{x}}{r^o}) = \tilde{g}(d, \frac{\bar{x}}{r^o}, r^o) = M(d, \frac{\bar{x}}{r^o}, r^o)G(d, \frac{\bar{x}}{r^o}, r^o) \\ \Gamma(d, \bar{x}) = \Gamma(d, r^o \frac{\bar{x}}{r^o}) = \tilde{\Gamma}(d, \frac{\bar{x}}{r^o}, r^o) = M(d, \frac{\bar{x}}{r^o}, r^o)\Gamma^*(d, \frac{\bar{x}}{r^o}, r^o) \end{cases}$$

But since g has at most two irreducible components, this would imply that for $d^o \notin \Delta$, $\mathcal{O}_{d^o}(\Sigma)$ would have at least a normal-isotropic component, contradicting Proposition 1.10(3) (page 14). Therefore, the equation $\Phi(d, \bar{k}) = 0$ defines a proper closed subset of \mathbb{C}^4 . This shows that we can take:

$$\Omega_2^0 = (\Omega_1^0 \cap \Omega_{\text{Iso}(\Sigma)}) \setminus \{(d^o, \bar{k}^o) : \Phi(d^o, \bar{k}^o) = 0\}$$

Then, for $(d^o, \bar{k}^o) \in \Omega_2^0$, each of the points \bar{x}_i^o , for $i = 1, \dots, \delta$, is associated with a unique non-normal isotropic point \bar{y}_i^o of Σ (recall that $d^o \in \Delta$, and so the irreducible components of $\mathcal{O}_{d^o}(\Sigma)$ are simple).

Let Ω_\perp be the open subset of $\mathbb{C} \times \mathbb{C}^3$ obtained by applying Lemma 2.4 (page 33) to the closed subset Σ_\perp whose existence is guaranteed by Lemma 2.1 (page 31). Recall that, by assumption (see Remark 2.6(2), page 35), Σ is not a sphere centered at the origin. Besides, let $\Theta = \mathbb{C}^3 \setminus \mathcal{L}_0$, where

$$\mathcal{L}_0 = \begin{cases} \emptyset & \text{if } \bar{0} \notin \Sigma \text{ or if } \bar{0} \in \text{Sing}(\Sigma) \\ \text{the normal line to } \Sigma \text{ at } \bar{0} & \text{otherwise.} \end{cases}$$

and set

$$\Omega_3^0 = \Omega_2^0 \cap \Omega_\perp \cap (\mathbb{C} \times \Theta).$$

Then for $(d^o, \bar{k}^o) \in \Omega_3^0$, the points $\bar{y}_i^o, i = 1, \dots, \delta$, are different. To prove this, note that if $\bar{y}_i = \bar{y}_j$, with $i \neq j$, then \bar{y}_i^o generates \bar{x}_i^o and \bar{x}_j^o . Thus, since $\bar{y}_i^o, \bar{x}_i^o, \bar{x}_j^o$ are all in the normal line to Σ at \bar{y}_i^o and in $\mathcal{L}_{\bar{k}^o}$, it follows that these two lines coincide. This means that $\bar{y}_i^o \in \Sigma_\perp$. Since $(d^o, \bar{k}^o) \in \Omega_\perp$, then (by Lemma 2.4) $\bar{y}_i^o \in \mathcal{L}_{\bar{k}^o} \cap \Omega_\perp$ implies that $\bar{y}_i^o = \bar{0}$, in contradiction with $\bar{k}^o \in \mathbb{C} \times \Theta$.

We will now show that it is possible to restrict the values of (d, \bar{k}) so that the points \bar{y}_i^o belong to the image of the parametrization P . Let Υ_2 be as in Lemma 1.24 (page 23), and let $\Omega_{\Upsilon_2} \subset \mathbb{C} \times \mathbb{C}^3$ be the open subset obtained applying Lemma 2.4 to $\Sigma \setminus \Upsilon_2$. Then take $\Omega_4^0 = \Omega_3^0 \cap \Omega_{\Upsilon_2}$.

Let us show that we can take $\Omega_0 = \Omega_4^0$. If $(d^o, \bar{k}^o) \in \Omega_4^0$, then for each of the points \bar{y}_i^o there are μ values $\bar{t}_{(i,j)}^o$ (with $i = 1, \dots, \delta, j = 1, \dots, m$) such that $P(\bar{t}_{(i,j)}^o) = \bar{y}_i^o$. Setting $\Psi_2^i(d^o, \bar{k}^o) = \{\bar{t}_{(i,j)}^o\}_{j=1, \dots, m}$, one has that

$$\Psi_2^P(d^o, \bar{k}^o) = \Psi_2^1(d^o, \bar{k}^o) \cup \dots \cup \Psi_2^\delta(d^o, \bar{k}^o)$$

and so the first part of claim (2) is proved. Furthermore:

- claim (a) holds because of the construction of Ω_3^0 .
- the structure of $\Psi_2(d^o, \bar{k}^o)$ in claims (b1) and (b2) holds because of the construction of Ω_4^0 .
- Claims (b3) and (b4) hold because of the construction of Ω_0^0, Ω_1^0 and Ω_2^0 .
- Claim (c) follows the construction of Ω_0^0 .

□

Remark 2.9. Note that, by the construction of Ω_0^0 in the proof of Theorem 2.8 (page 35), if $(d^o, \bar{k}^o) \in \Omega_0$, then $g(d^o, \bar{x}) = 0$ is the equation of $\mathcal{O}_{d^o}(\Sigma)$.

2.2 Elimination and auxiliary polynomials

To continue with our strategy, we proceed to eliminate the variables (l, r, \bar{x}) in the Parametric Offset-Line System $\mathfrak{S}_2^P(d, \bar{k})$ (page 31). This elimination process leads us to consider the following system of equations:

$$\mathfrak{S}_3^P(d, \bar{k}) \equiv \begin{cases} s_1(d, \bar{k}, \bar{t}) := h(\bar{t})(k_2 P_3 - k_3 P_2)^2 - d^2 P_0(\bar{t})^2 (k_2 n_3 - k_3 n_2)^2 = 0, \\ s_2(d, \bar{k}, \bar{t}) := h(\bar{t})(k_1 P_3 - k_3 P_1)^2 - d^2 P_0(\bar{t})^2 (k_1 n_3 - k_3 n_1)^2 = 0, \\ s_3(d, \bar{k}, \bar{t}) := h(\bar{t})(k_1 P_2 - k_2 P_1)^2 - d^2 P_0(\bar{t})^2 (k_1 n_2 - k_2 n_1)^2 = 0. \end{cases} \quad (12)$$

We will refer to this as the **Affine Auxiliary System**.

We recall that $P = \left(\frac{P_1}{P_0}, \frac{P_2}{P_0}, \frac{P_3}{P_0} \right)$, $\bar{k} = (k_1, k_2, k_3)$, $\bar{n} = (n_1, n_2, n_3)$ and $h(\bar{t}) = n_1(t)^2 + n_2(t)^2 + n_3(t)^2$. Along with the polynomials s_1, s_2, s_3 introduced in the above system, we will also need to consider the following polynomial:

$$s_0(\bar{k}, \bar{t}) = k_1(P_2n_3 - P_3n_2) - k_2(P_1n_3 - P_3n_1) + k_3(P_1n_2 - P_2n_1)$$

The geometrical meaning of s_0 is clear when one expresses it as a determinant, as follows:

$$s_0(\bar{k}, \bar{t}) = \det \begin{pmatrix} k_1 & k_2 & k_3 \\ P_1 & P_2 & P_3 \\ n_1 & n_2 & n_3 \end{pmatrix}. \quad (13)$$

We will introduce some additional notation to simplify the expression of the polynomials s_i for $i = 1, 2, 3$. More precisely, we **denote**:

$$\begin{cases} M_1(\bar{k}, \bar{t}) = k_2P_3 - k_3P_2, & G_1(\bar{k}, \bar{t}) = k_2n_3 - k_3n_2, \\ M_2(\bar{k}, \bar{t}) = k_3P_1 - k_1P_3, & G_2(\bar{k}, \bar{t}) = k_3n_1 - k_1n_3, \\ M_3(\bar{k}, \bar{t}) = k_1P_2 - k_2P_1, & G_3(\bar{k}, \bar{t}) = k_1n_2 - k_2n_1. \end{cases} \quad (14)$$

With this notation one has

$$s_i(d, \bar{k}, \bar{t}) = h(\bar{t})M_i^2(\bar{k}, \bar{t}) - d^2P_0(\bar{t})^2G_i^2(\bar{k}, \bar{t}) \text{ for } i = 1, 2, 3.$$

Note also that

$$\begin{cases} (M_1, M_2, M_3)(\bar{k}, \bar{t}) = \bar{k} \wedge (P_1(\bar{t}), P_2(\bar{t}), P_3(\bar{t})) \\ (G_1, G_2, G_3)(\bar{k}, \bar{t}) = \bar{k} \wedge \bar{n}(\bar{t}). \end{cases} \quad (15)$$

Let

$$I_2^P(d) = \langle b^P, \text{nor}_{(1,2)}^P, \text{nor}_{(1,3)}^P, \text{nor}_{(2,3)}^P, w^P, \ell_1, \ell_2, \ell_3 \rangle \subset \mathbb{C}[d, \bar{k}, l, r, \bar{t}, \bar{x}]$$

be the ideal generated by the polynomials that define the Parametric Offset-Line System $\mathfrak{S}_2^P(d, \bar{k})$. We consider the projection associated with the elimination:

$$\pi_{(2,1)} : \mathbb{C} \times \mathbb{C}^3 \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}^2 \times \mathbb{C}^3 \mapsto \mathbb{C} \times \mathbb{C}^3 \times \mathbb{C}^2$$

given by

$$\pi_{(2,1)}(d, \bar{k}, l, r, \bar{t}, \bar{x}) = (d, \bar{k}, \bar{t})$$

The next lemma relates the polynomials $s_0, \dots, s_3 \in \mathbb{C}[d, \bar{k}, \bar{t}]$ in System $\mathfrak{S}_3(d, \bar{k})$ with the elimination process. We **denote** by $\tilde{I}_2^P(d)$ the elimination ideal $I_2^P(d) \cap \mathbb{C}[d, \bar{k}, \bar{t}]$. For $(d^o, \bar{k}^o) \in \mathbb{C} \times \mathbb{C}^3$, the set of solutions of the Parametric Offset-Line system is denoted by $\Psi_2^P(d^o, \bar{k}^o)$, and the set of solutions of $\mathfrak{S}_3^P(d^o, \bar{k}^o)$ is denoted by $\Psi_3^P(d^o, \bar{k}^o)$. Note that $\Psi_2^P(d^o, \bar{k}^o) = \mathbf{V}(I_2^P(d))$.

Lemma 2.10. $s_i \in \tilde{I}_2^P(d)$ for $i = 0, \dots, 3$. In particular, if $(l^o, r^o, \bar{t}^o, \bar{x}^o) \in \Psi_2^P(d^o, \bar{k}^o)$, then $\bar{t}^o \in \Psi_3^P(d^o, \bar{k}^o)$.

Proof. The polynomials s_i can be expressed as follows:

$$s_i = c_1^{(i)} b^P + c_2^{(i)} \text{nor}_{(1,2)}^P + c_3^{(i)} \text{nor}_{(1,3)}^P + c_4^{(i)} \text{nor}_{(2,3)}^P + c_5^{(i)} w^P + c_6^{(i)} \ell_1 + c_7^{(i)} \ell_2 + c_8^{(i)} \ell_3$$

where $c_j^{(i)} \in \mathbb{C}[d, \bar{k}, l, r, \bar{t}, \bar{x}]$ for $i = 0, \dots, 3$, $j = 1, \dots, 8$. This polynomials (obtained with the CAS Singular [?]) can be found in Appendix 3.4 (page 81). \square

The next step appears naturally to be the converse analysis: which are the $\bar{t}^o \in \mathfrak{S}_3^P(d^o, \bar{k}^o)$ that can be extended to a solution $(l^o, r^o, \bar{t}^o, \bar{x}^o) \in \Psi_2^P(d^o, \bar{k}^o)$? In order to describe them, we need some notation and a lemma. Let \mathcal{A} denote the set of values $\bar{t}^o \in \mathbb{C}^2$ such that:

$$\begin{cases} P_0(\bar{t}^o)h(\bar{t}^o)(P_2(\bar{t}^o)n_3(\bar{t}^o) - P_3(\bar{t}^o)n_2(\bar{t}^o)) \neq 0 \\ \text{or} \\ P_0(\bar{t}^o)h(\bar{t}^o)(P_1(\bar{t}^o)n_3(\bar{t}^o) - P_3(\bar{t}^o)n_1(\bar{t}^o)) \neq 0 \\ \text{or} \\ P_0(\bar{t}^o)h(\bar{t}^o)(P_1(\bar{t}^o)n_2(\bar{t}^o) - P_2(\bar{t}^o)n_1(\bar{t}^o)) \neq 0 \end{cases} \quad (16)$$

Now we can describe which solutions of $\bar{t}^o \in \mathfrak{S}_3^P(d^o, \bar{k}^o)$ can be extended.

Proposition 2.11. Let Ω_0 be as in Theorem 2.8 (page 35), $(d^o, \bar{k}^o) \in \Omega_0$ and $\bar{t}^o \in \Psi_3^P(d^o, \bar{k}^o)$. Then the following holds:

(a) There exists $\lambda^o \in \mathbb{C}^\times$ such that:

$$\bar{k}^o \wedge (P_1(\bar{t}^o), P_2(\bar{t}^o), P_3(\bar{t}^o)) = \lambda^o (\bar{k}^o \wedge \bar{n}(\bar{t}^o)).$$

That is,

$$M_i(\bar{k}^o, \bar{t}^o) = \lambda^o G_i(\bar{k}^o, \bar{t}^o) \text{ for } i = 1, 2, 3.$$

(b) If $\bar{t}^o \in \mathcal{A}$, then $s_0(d^o, k^o, \bar{t}^o) = 0$.

(c) $(d^o, \bar{k}^o, \bar{t}^o) \in \pi_{(2,1)}(\Psi_2^P(d^o, \bar{k}^o))$ if and only if $\bar{t}^o \in \mathcal{A}$.

In particular,

$$m\delta = \# (\mathcal{A} \cap \Psi_3^P(d^o, \bar{k}^o)).$$

Recall that m is the tracing index of P , and δ is the total degree w.r.t \bar{x} of the generic offset equation.

Proof.

- (a) To prove the existence of λ^o , notice that $\bar{t}^o \in \Psi_3^P(d^o, \bar{k}^o)$ implies:

$$h(\bar{t}^o)M_i^2(\bar{k}^o, \bar{t}^o) = (d^o)^2 P_0(\bar{t}^o)^2 G_i^2(\bar{k}^o, \bar{t}^o) \text{ for } i = 1, 2, 3.$$

Since $\bar{t}^o \in \mathcal{A}$, $h(\bar{t}^o) \neq 0$. Therefore one concludes that there exist ϵ_i , with $\epsilon_i^2 = 1$, such that

$$M_i(\bar{k}^o, \bar{t}^o) = \epsilon_i \frac{d^o P_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}} G_i(\bar{k}^o, \bar{t}^o) \text{ for } i = 1, 2, 3.$$

Since there are three of them, two of the ϵ_i must coincide. We will show that the third one must coincide as well. That is, we will show that either $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, or $\epsilon_1 = \epsilon_2 = \epsilon_3 = -1$ holds. We will study one particular case, the other possible combinations can be treated similarly. Let us suppose, e.g., that $\epsilon_1 = \epsilon_2 = 1$. Then:

$$\begin{cases} k_2^o P_3(\bar{t}^o) - k_3^o P_2(\bar{t}^o) = \frac{d^o P_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}} (k_2^o n_3(\bar{t}^o) - k_3^o n_2(\bar{t}^o)) \\ k_3^o P_1(\bar{t}^o) - k_1^o P_3(\bar{t}^o) = \frac{d^o P_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}} (k_3^o n_1(\bar{t}^o) - k_1^o n_3(\bar{t}^o)) \end{cases}$$

Multiplying the first equation by k_1^o and the second by k_2^o , and subtracting one has:

$$k_3^o (k_1^o P_2(\bar{t}^o) - k_2^o P_1(\bar{t}^o)) = k_3^o \frac{d^o P_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}} (k_1^o n_2(\bar{t}^o) - k_2^o n_1(\bar{t}^o))$$

Since $(d^o, \bar{k}^o) \in \Omega_0$, we have $k_3^o \neq 0$ (see Theorem 2.8(c), page 35). Thus, we have shown that

$$k_1^o P_2(\bar{t}^o) - k_2^o P_1(\bar{t}^o) = \frac{d^o P_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}} (k_1^o n_2(\bar{t}^o) - k_2^o n_1(\bar{t}^o))$$

and so $\epsilon_3 = 1$. Therefore, $\lambda^o = \frac{d^o P_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}}$, and it is non-zero because $\bar{t}^o \in \mathcal{A}$.

- (b) From the identity in (a) it follows immediately that \bar{k}^o , $(P_1(\bar{t}^o), P_2(\bar{t}^o), P_3(\bar{t}^o))$ and $\bar{n}(\bar{t}^o)$ are coplanar vectors. Thus, $s_0(d^o, \bar{k}^o, \bar{t}^o) = 0$ (recall the geometric interpretation of s_0 in equation 13, page 39).
- (c) If $(d^o, \bar{k}^o, \bar{t}^o) \in \pi_{(2,1)}(\Psi_2^P(d^o, \bar{k}^o))$, then $P_0(\bar{t}^o)\beta(\bar{t}^o)h(\bar{t}^o) \neq 0$ follows from equation w^P in the Parametric Offset-Line System 11 (page 31). Besides,

$$(P_2 n_3 - P_3 n_2)(\bar{t}^o) = (P_1 n_3 - P_3 n_1)(\bar{t}^o) = (P_1 n_2 - P_2 n_1)(\bar{t}^o) = 0$$

is impossible because of Theorem 2.8(a) (page 35). Thus $\bar{t}^o \in \mathcal{A}$.

Conversely, let us suppose that $\bar{t}^o \in \mathcal{A}$. More precisely, let us suppose w.l.o.g. that

$$P_0(\bar{t}^o)h(\bar{t}^o)(P_2(\bar{t}^o)n_3(\bar{t}^o) - P_3(\bar{t}^o)n_2(\bar{t}^o)) \neq 0.$$

The other cases can be proved in a similar way. First we note that

$$G_1(\bar{k}^o, \bar{t}^o) = k_3^o n_2(\bar{t}^o) - k_2^o n_3(\bar{t}^o) \neq 0.$$

since, using that $s_1(d^o, \bar{k}^o, \bar{t}^o) = 0$ and $h(\bar{t}^o) \neq 0$, one has that

$$k_2^o P_3(\bar{t}^o) - k_3^o P_2(\bar{t}^o) = 0.$$

Then, from the system:

$$\begin{cases} k_2^o n_3(\bar{t}^o) - k_3^o n_2(\bar{t}^o) = 0 \\ k_2^o P_3(\bar{t}^o) - k_3^o P_2(\bar{t}^o) = 0 \end{cases}$$

and the fact that $k_2^o k_3^o \neq 0$ (again, this is Theorem 2.8(c)), one deduces that

$$P_2(\bar{t}^o) n_3(\bar{t}^o) - P_3(\bar{t}^o) n_2(\bar{t}^o) = 0,$$

that is a contradiction. Thus, we can define

$$r^o = \frac{1}{P_0(\bar{t}^o) \beta(\bar{t}^o) h(\bar{t}^o)}, \text{ and } l^o = \frac{P_3(\bar{t}^o) n_2(\bar{t}^o) - P_2(\bar{t}^o) n_3(\bar{t}^o)}{-P_0(\bar{t}^o) G_1(\bar{k}^o, \bar{t}^o)}$$

We also define $\bar{x}^o = l^o \bar{k}^o$. We claim that $(l^o, r^o, \bar{t}^o, \bar{x}^o) \in \Psi_2^P(d^o, \bar{k}^o)$, and therefore $(d^o, \bar{k}^o, \bar{t}^o) \in \pi_{(2,1)}(\Psi_2^P(d^o, \bar{k}^o))$. To prove our claim we substitute $(l^o, r^o, \bar{t}^o, \bar{x}^o)$ in the equations of the Parametric Offset-Line System 11 (page 31), and we check that all of them vanish. The vanishing of $w^P(r^o, \bar{t}^o)$ and $\ell_i(\bar{k}^o, l^o, \bar{x}^o)$ for $i = 1, 2, 3$ is a trivial consequence of the definitions. Substitution in $\text{nor}_{(2,3)}^P$ leads to a polynomial whose numerator vanishes immediately. Substituting in $\text{nor}_{(1,2)}^P$ (resp. in $\text{nor}_{(1,3)}^P$) one obtains:

$$\text{nor}_{(1,2)}^P(\bar{t}^o, \bar{x}^o) = \frac{n_2(\bar{t}^o) s_0(\bar{k}^o, \bar{t}^o)}{n_2 \bar{t}^o k_3^o - n_3(\bar{t}^o) k_2^o} = 0,$$

(respectively

$$\text{nor}_{(1,3)}^P(\bar{t}^o, \bar{x}^o) = \frac{n_3(\bar{t}^o) s_0(\bar{k}^o, \bar{t}^o)}{n_2 \bar{t}^o k_3^o - n_3(\bar{t}^o) k_2^o} = 0),$$

where both equations hold because of part (a). Finally, substituting in $b^P(d^o, \bar{t}^o, \bar{x}^o)$ one has:

$$b^P(d^o, \bar{t}^o, \bar{x}^o) = \frac{s_2(d^o, \bar{k}^o, \bar{t}^o) + \phi_1(\bar{k}^o, \bar{t}^o) s_0(\bar{k}^o, \bar{t}^o)}{(n_2 \bar{t}^o k_3^o - n_3(\bar{t}^o) k_2^o)^2} = 0 \quad (17)$$

with $\phi_1(\bar{k}, \bar{t}) = k_2 n_1 P_3 + k_2 n_3 P_1 - k_3 n_1 P_2 - k_3 n_2 P_1 - k_1 n_3 P_2 + k_1 n_2 P_3$. Equation 17 holds because of part (a) and because $s_2(d^o, \bar{k}^o, \bar{t}^o) = 0$.

The claim that

$$m\delta = \#(\mathcal{A} \cap \Psi_3^P(d^o, \bar{k}^o))$$

follows easily from Theorem 2.8(b) (page 35) and the above result (c). This shows that, for $(d^o, \bar{k}^o) \in \Omega_0$, there is a bijection (under $\pi_{(2,1)}$) between the points of $\Psi_2^P(d^o, \bar{k}^o)$ and the points in $\mathcal{A} \cap \Psi_3^P(d^o, \bar{k}^o)$. This finishes the proof of the proposition. \square

Remark 2.12. *In the proof of Proposition 2.11 (page 2.11) we have seen that there is a vector equality:*

$$\bar{M}(\bar{k}^o, \bar{t}^o) = \epsilon \frac{d^o P_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}} \bar{G}(\bar{k}^o, \bar{t}^o) \text{ for } i = 1, 2, 3.$$

where $\bar{M} = (M_1, M_2, M_3)$, $\bar{G} = (G_1, G_2, G_3)$ and $\epsilon = \pm 1$. In the next lemma we will see that the value of $\epsilon = 1$ determines the sign that appears in the offsetting construction. More precisely, in the proof of Proposition 2.11 we have seen that if $\bar{y}^o = P(\bar{t}^o)$, and

$$P_0(\bar{t}^o)h(\bar{t}^o)(P_2(\bar{t}^o)n_3(\bar{t}^o) - P_3(\bar{t}^o)n_2(\bar{t}^o)) \neq 0.$$

then it holds that

$$k_2^o n_3(\bar{t}^o) - k_3^o n_2(\bar{t}^o) \neq 0 \text{ and } k_2^o P_3(\bar{t}^o) - k_3^o P_2(\bar{t}^o) \neq 0.$$

Furthermore, the point \bar{x}^o , constructed as follows

$$\bar{x}^o = \frac{P_3(\bar{t}^o)n_2(\bar{t}^o) - P_2(\bar{t}^o)n_3(\bar{t}^o)}{-P_0(\bar{t}^o)G_1(\bar{k}^o, \bar{t}^o)} \bar{k}^o, \quad (18)$$

is the point in $\mathcal{O}_{d^o}(\Sigma) \cap \mathcal{L}_{\bar{k}^o}$ associated with \bar{y}^o . Thus, one has:

$$\bar{x}^o = \bar{y}^o + \epsilon' \frac{d^o \nabla f(\bar{y}^o)}{\sqrt{h_{\text{imp}}(\bar{y}^o)}}.$$

where $\epsilon' = \pm 1$.

Lemma 2.13. *With the notation of Remark 2.12, it holds that $\epsilon = \epsilon'$.*

Proof. From the Equations

$$M_2(\bar{k}^o, \bar{t}^o) = \epsilon \frac{d^o P_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}} G_2(\bar{k}^o, \bar{t}^o) \text{ and } M_3(\bar{k}^o, \bar{t}^o) = \epsilon \frac{d^o P_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}} G_3(\bar{k}^o, \bar{t}^o),$$

multiplying the first equation by $n_2(\bar{t}^o)$, the second by $n_3(\bar{t}^o)$ and adding the results, one has:

$$-G_1(\bar{k}^o, \bar{t}^o)P_1(\bar{t}^o) - k_1^o(P_3n_2 - P_2n_3)(\bar{t}^o) = \epsilon n_1(\bar{t}^o) \frac{d^o P_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}} G_1(\bar{k}^o, \bar{t}^o)$$

Using Equation 18 in Remark 2.12, this is:

$$-G_1(\bar{k}^o, \bar{t}^o)P_1(\bar{t}^o) + x_1^o G_1(\bar{k}^o, \bar{t}^o)P_0(\bar{t}^o) = \epsilon n_1(\bar{t}^o) \frac{d^o P_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}} G_1(\bar{k}^o, \bar{t}^o).$$

Dividing by $G_1(\bar{k}^o, \bar{t}^o)P_0(\bar{t}^o)$:

$$-\frac{P_1(\bar{t}^o)}{P_0(\bar{t}^o)} + x_1^o = \epsilon \frac{d^o n_1(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}},$$

and finally

$$x_1^o = \frac{P_1(\bar{t}^o)}{P_0(\bar{t}^o)} + \epsilon \frac{d^o n_1(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}}.$$

Similar results are obtained for x_2^o and x_3^o . Thus we have proved that $\epsilon' = \epsilon$. \square

2.3 Fake points

Using Proposition 2.11 (page 40) we can now define the set of fake points associated with this problem.

Definition 2.14. *A point $\bar{t}^o \in \mathbb{C}^2$ is a fake point if*

$$\begin{cases} P_0(\bar{t}^o)h(\bar{t}^o)(P_2(\bar{t}^o)n_3(\bar{t}^o) - P_3(\bar{t}^o)n_2(\bar{t}^o)) = 0 \\ \quad \text{and} \\ P_0(\bar{t}^o)h(\bar{t}^o)(P_1(\bar{t}^o)n_3(\bar{t}^o) - P_3(\bar{t}^o)n_1(\bar{t}^o)) = 0 \\ \quad \text{and} \\ P_0(\bar{t}^o)h(\bar{t}^o)(P_1(\bar{t}^o)n_2(\bar{t}^o) - P_2(\bar{t}^o)n_1(\bar{t}^o)) = 0 \end{cases}$$

Equivalently,

$$P_0(\bar{t}^o)h(\bar{t}^o) = 0 \text{ or } (P_1(\bar{t}^o), P_2(\bar{t}^o), P_3(\bar{t}^o)) \wedge \bar{n}(\bar{t}^o) = \bar{0} \quad (19)$$

The set of fake points will be denoted by \mathcal{F} .

Definition 2.15. *Let Ω_0 be as in Theorem 2.8 (page 35) and let Ω be any open subset of Ω_0 . The set of invariant solutions of $\mathfrak{S}_3^P(d, \bar{k})$ w.r.t Ω . is defined as the set:*

$$\mathcal{I}_3^P(\Omega) = \bigcap_{(d^o, \bar{k}^o) \in \Omega} \Psi_3^P(d^o, \bar{k}^o)$$

Remark 2.16. *Note that if $\bar{t}^o \in \mathcal{F}$, we do not assume that $\bar{t}^o \in \Psi_3^P(d^o, \bar{k}^o)$ for some $(d^o, \bar{k}^o) \in \mathbb{C} \times \mathbb{C}^3$.*

We have introduced the fake points starting from the notion non-extendable solutions of $\mathfrak{S}_3^P(d^o, \bar{k}^o)$. Another point of view is to define fake points as the *invariant solutions* of $\mathfrak{S}_3^P(d, \bar{k})$. First we will define what we mean by invariant in this context, and then we will show that, in a certain open subset of values (d, \bar{k}) , both notions actually coincide.

To prove the equivalence between the notions of fake points and invariant points we need to further restrict the set of values of (d, \bar{k}) . The following lemma gives the required restrictions.

Lemma 2.17. *Let Ω_0 be the open set in Theorem 2.8. There exists an open non-empty $\Omega_1 \subset \Omega_0$ such that if $(d^o, \bar{k}^o) \in \Omega_1$, then*

(1) \bar{k}^o is not isotropic.

(2) d^o is not a critical distance of Σ (see Corollary 2.3 in page 32).

(3) The system

$$\{P_0(\bar{t}) = M_1(\bar{k}^o, 1, \bar{t}) = M_2(\bar{k}^o, 1, \bar{t}) = M_3(\bar{k}^o, 1, \bar{t}) = 0\} \quad (20)$$

has no solutions unless $P_0(\bar{t}^o) = P_1(\bar{t}^o) = P_2(\bar{t}^o) = P_3(\bar{t}^o) = 0$.

Proof.

- (1) Set $\Omega_1^1 = \Omega_0 \cap (\mathbb{C} \times \mathfrak{Q})$, where $\mathfrak{Q} = \{\bar{k}^o / (k_1^o)^2 + (k_2^o)^2 + (k_3^o)^2 = 0\}$ is the cone of isotropy in \bar{k} .
- (2) Let $\Upsilon(\Sigma^\perp)$ is the set of critical distances of Σ (defined in page 32), and set $\Omega_1^2 = \Omega_1^1 \cap (\Upsilon(\Sigma^\perp) \times \mathbb{C}^3)$.
- (3) First we will show that the set of values $\bar{k}^o \neq \bar{0}$ for which the System 20 has a solution is contained in an at most two-dimensional closed subset $\mathfrak{R} \subset \mathbb{C}^3$. If P_0 is constant the result is trivial. Assuming that P_0 is not constant, let \mathcal{C}_0 be the affine curve defined by P_0 , and let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ be the varieties defined by P_1, P_2, P_3 respectively. Let $\mathcal{J}_{P_1} \subset \mathbb{C}[\bar{k}, \bar{t}, v]$ be the ideal defined as follows:

$$\mathcal{J}_{P_1} = \langle P_0, k_2 P_3 - k_3 P_2, k_1 P_3 - k_3 P_1, k_1 P_2 - k_2 P_1, v P_1 - 1 \rangle,$$

and let $\mathbf{V}(\mathcal{J}_{P_1}) \subset \mathbb{C}^3 \times \mathbb{C}^2 \times \mathbb{C}$ be the solution set of this ideal. Consider the projections defined by:

$$\begin{cases} \pi_1(\bar{k}, \bar{t}, v) = \bar{k} \\ \pi_2(\bar{k}, \bar{t}, v) = \bar{t} \end{cases}$$

Let A_0 be an irreducible component of $\mathbf{V}(\mathcal{J}_{P_1})$, and let $(\bar{k}^o, \bar{t}^o, v^o) \in A_0$. Then the points in $\pi_2^{-1}(\pi_2(\bar{k}^o, \bar{t}^o, v^o))$ are the solutions of the following system:

$$\begin{cases} \bar{t} = \bar{t}^o, \\ M_1(\bar{k}, 1, \bar{t}^o) = M_2(\bar{k}, 1, \bar{t}^o) = M_3(\bar{k}, 1, \bar{t}^o) = 0, \\ vP_1(\bar{t}^o) - 1 = 0 \end{cases}$$

The dimension of the set of solutions is 1. On the other hand, $\pi_2(A_0) \subset \mathcal{C}_0$ implies that $\dim(\pi_2(A_0)) \leq 1$. Thus, using Lemma 1.4 (page 12), one has that $\dim(\mathbf{V}(\mathcal{J}_{P_1})) \leq 2$. Thus, $\dim(\pi_1(\mathbf{V}(\mathcal{J}_{P_1}))^*) \leq 2$. Now, defining \mathcal{J}_{P_2} and \mathcal{J}_{P_3} in a similar way (that is, replacing the equation $vP_1(\bar{t}^o) - 1 = 0$ by $vP_2(\bar{t}^o) - 1 = 0$ and $vP_3(\bar{t}^o) - 1 = 0$ respectively), we set:

$$\mathfrak{R} = \bigcup_{i=1,2,3} \pi_1(\mathbf{V}(\mathcal{J}_{P_i}))^*$$

Now let $\Omega_1^3 = \Omega_1^2 \cap (\mathbb{C} \times \mathfrak{R})$.

The above construction shows that $\Omega_1 = \Omega_1^3$ satisfies the required properties. \square

Now we can prove the announced equivalence between the notions of fake points and invariant points.

Proposition 2.18. *Let Ω_1 be as in Lemma 2.17 (page 45). If Ω is a non-empty open subset of Ω_1 , then it holds that:*

$$\mathcal{I}_3^P(\Omega) = \mathcal{F} \cap \left(\bigcup_{(d^o, \bar{k}^o) \in \Omega} \Psi_3^P(d^o, \bar{k}^o) \right).$$

Proof. Let $\bar{t}^o \in \mathcal{I}_3^P(\Omega)$. Then $s_i(d^o, \bar{k}^o, \bar{t}^o) = 0$ for $i = 1, 2, 3$ and any $(d^o, \bar{k}^o) \in \Omega$. Thus $\bar{t}^o \in \bigcup_{(d^o, \bar{k}^o) \in \Omega} \Psi_3^P(d^o, \bar{k}^o)$. Furthermore, considering s_i as polynomials in $\mathbb{C}[\bar{t}][d, \bar{k}]$, it follows that \bar{t}^o must be a solution of:

$$\begin{cases} h(\bar{t})P_1(\bar{t}) = h(\bar{t})P_2(\bar{t}) = h(\bar{t})P_3(\bar{t}) = 0 \\ P_0(\bar{t})n_1(\bar{t}) = P_0(\bar{t})n_2(\bar{t}) = P_0(\bar{t})n_3(\bar{t}) = 0 \end{cases}$$

It follows that $h(\bar{t}^o)P_0(\bar{t}^o) = 0$, and so $\bar{t}^o \in \mathcal{F}$. In fact, if we suppose $h(\bar{t}^o)P_0(\bar{t}^o) \neq 0$, then from $P_0(\bar{t}^o) \neq 0$ one gets $\bar{n}(\bar{t}^o) = 0$, and so $h(\bar{t}^o) = 0$, a contradiction.

Conversely, let $\bar{t}^o \in \mathcal{F} \cap \left(\bigcup_{(d^o, \bar{k}^o) \in \Omega} \Psi_3^P(d^o, \bar{k}^o) \right)$. Then:

1. If $P_0(\bar{t}^o) = h(\bar{t}^o) = 0$, then $s_i(d, \bar{k}, \bar{t}^o) = 0$ identically in (d, \bar{k}) for $i = 1, 2, 3$, and so $\bar{t}^o \in \mathcal{I}_3^P(\Omega)$.

2. If $P_0(\bar{t}^o) \neq 0$ and $h(\bar{t}^o) = 0$, then since $\bar{t}^o \in \Psi_3^P(d^o, \bar{k}^o)$ for some $(d^o, \bar{k}^o) \in \Omega$, one has the following two possibilities:
 - (a) $\bar{n}(\bar{t}^o)$ is isotropic and parallel to \bar{k}^o . This is impossible because of the construction of Ω_1 (see Lemma 2.17(1), page 45).
 - (b) $\bar{n}(\bar{t}^o) = \bar{0}$. In this case, again $s_i(d, \bar{k}, \bar{t}^o) = 0$ identically in (d, \bar{k}) for $i = 1, 2, 3$, and so $\bar{t}^o \in \mathcal{I}_3^P(\Omega)$.
3. Let us suppose that $P_0(\bar{t}^o) = 0$ and $h(\bar{t}^o) \neq 0$. Then, since $\bar{t}^o \in \Psi_3^P(d^o, \bar{k}^o)$ for some $(d^o, \bar{k}^o) \in \Omega$, one has that \bar{t}^o is a solution of:

$$P_0(\bar{t}) = 0, \quad M_1(\bar{t}, \bar{k}^o) = M_2(\bar{t}, \bar{k}^o) = M_3(\bar{t}, \bar{k}^o) = 0,$$

Thus, two cases are possible:

- (a) $(P_1, P_2, P_3)(\bar{t}^o) = \bar{0}$. In this case, $s_i(d, \bar{k}, \bar{t}^o) = 0$ identically in (d, \bar{k}) for $i = 1, 2, 3$, and so $\bar{t}^o \in \mathcal{I}_3^P(\Omega)$.
 - (b) $(P_1, P_2, P_3)(\bar{t}^o)$ is non-zero. This contradicts the construction of Ω_1 in Lemma 2.17(3).
4. Finally, let us suppose that $P_0(\bar{t}^o)h(\bar{t}^o) \neq 0$. Then it follows that the point $P(\bar{t}^o)$ is well defined, and it belongs to Σ_\perp^* (recall that Σ_\perp was introduced in Lemma 2.1, page 31). Thus d^o would be one of the critical distances, and this contradicts the construction of Ω_0 in Lemma 2.17(2). \square

3 Total Degree Formula for Parametric Surfaces

According to Proposition 2.11 (page 40), if $(d^o, \bar{k}^o) \in \Omega_0$ it holds that

$$m\delta = \#(\mathcal{A} \cap \Psi_3^P(d^o, \bar{k}^o)).$$

Recall that m is the tracing index of P , and δ is the total degree w.r.t \bar{x} of the generic offset equation. Moreover, \mathcal{A} was introduced in Equation 16 (page 40), and $\Psi_3^P(d^o, \bar{k}^o)$ was also introduced in page 39, as the solution set of System 12 (page 38). In this section, we will derive a formula for the total degree δ , using the tools in Section 1.3 (page 27) to analyze the intersection $\mathcal{A} \cap \Psi_3^P(d^o, \bar{k}^o)$.

In order to do this:

- in Subsection 3.1 we will consider the projective closure of the auxiliary curves introduced in the preceding section. This in turn, requires as a first step the projectivization of the parametrization P . At the end of the subsection we introduce the Projective Auxiliary System 30 (page 52), which will play a key rôle in the degree formula.

- Subsection 3.2. (page 52) is devoted to the study of the invariant solutions of the Projective Auxiliary System, connecting them with the corresponding affine notions in Section 2.2. A crucial step in our strategy concerns the multiplicity of intersection of the auxiliary curves at their non-invariant points of intersection.
- In Subsection 3.3 (page 61) we will prove (in Proposition 3.20, page 62) that the value of that multiplicity of intersection has the required property for the use of generalized resultants (according to Lemma 1.33, page 28) .
- After this is done, everything is ready for the proof of the degree formula, which is the topic of Subsection 3.4 (page 71). The formula appears in Theorem 3.22 (page 73).

3.1 Projectivization of the parametrization and auxiliary curves

Let P be the parametrization of Σ , introduced in Equation (6). If we homogenize the components of P w.r.t. a new variable t_0 , multiplying both the numerators and denominators if necessary by a suitable power of t_0 we arrive at an expression of the form:

$$P_h(\bar{t}_h) = \left(\frac{X(\bar{t}_h)}{W(\bar{t}_h)}, \frac{Y(\bar{t}_h)}{W(\bar{t}_h)}, \frac{Z(\bar{t}_h)}{W(\bar{t}_h)} \right) \quad (21)$$

where $\bar{t}_h = (t_0 : t_1 : t_2)$, and $X, Y, Z, W \in \mathbb{C}[\bar{t}_h]$ are homogeneous polynomials of the same degree d_P , for which $\gcd(X, Y, Z, W) = 1$ holds. This P_h will be called the projectivization of P .

Remark 3.1. *Note that those projective values of \bar{t}_h of the form $(0 : a : b)$ correspond to points at infinity in the parameter plane.*

In Section 1.2 (page 22) we defined $\bar{n} = (n_1, n_2, n_3)$, the associated normal vector to P . A similar construction, applied to P_h , leads to a normal vector $N = (N_1, N_2, N_3)$, where N_i are homogeneous polynomials in \bar{t}_h of the same degree, such that $\gcd(N_1, N_2, N_3) = 1$. This vector N will be called the **associated homogeneous normal vector** to P_h . The homogeneous polynomial H defined by

$$H(\bar{t}_h) = (N_1(\bar{t}_h))^2 + (N_2(\bar{t}_h))^2 + (N_3(\bar{t}_h))^2$$

is the **parametric projective normal-hodograph** of the parametrization P_h .

Remark 3.2. *The polynomials N_i are, up to multiplication by a power of t_0 , the homogenization of the components of \bar{n} w.r.t. t_0 . However, since $\gcd(N_1, N_2, N_3) = 1$, at least one of the components N_i ($i = 1, 2, 3$) is not divisible by t_0 . Besides, note that if two components N_i, N_j , with $i \neq j$, are divisible by t_0 , then H is not.*

Lemma 3.3.

1. If W does not depend on t_0 , then at least one of the polynomials X, Y, Z must depend on t_0 .
2. If W does not depend on t_0 , and there is exactly one of the polynomials X, Y, Z depending on t_0 , then the surface is a cylinder with its axis parallel to the direction of the component with numerator depending on t_0 .

Proof

1. Otherwise, the rank of the jacobian matrix of P would be less than two. To see this, let us suppose that X, Y, Z, W depend only on t_1, t_2 . Let $\partial_i P_h$ be the vector obtained as the partial derivative of P_h w.r.t. t_i , that is;

$$\partial_i P_h = \left(\frac{X_i W - X W_i}{W^2}, \frac{Y_i W - Y W_i}{W^2}, \frac{Z_i W - Z W_i}{W^2} \right)$$

where X_i, Y_i, Z_i, W_i denotes the partial derivative of X, Y, Z, W w.r.t. t_i . Using Euler's formula, and taking into account that the polynomials X, Y, Z, W have the same degree n , one has that $t_1 \partial_1 P_h = -t_2 \partial_2 P_h$. Substituting $t_0 = 1$, we see that the rank of the jacobian of P would be less than 2.

2. Assume w.l.o.g. that X, Y do not depend on t_0 , but Z does. The rational map

$$\phi(\bar{t}) = \left(\frac{X(\bar{t})}{W(\bar{t})}, \frac{Y(\bar{t})}{W(\bar{t})} \right)$$

has rank one, because X, Y, W are homogeneous polynomials in \bar{t} of the same degree. Thus, ϕ parametrizes a curve \mathcal{C} in the (y_1, y_2) -plane. Let $\text{Cyl}(\mathcal{C})$ be the cylinder over \mathcal{C} with axis parallel to the y_3 -axis. The points of the form $(\phi(\bar{t}^o), y_3^o)$, with $W(\bar{t}^o) \neq 0$, are dense in $\text{Cyl}(\mathcal{C})$. Given one of these points, let t_0^o be any solution of the equation (in t_0):

$$Z(\bar{t}^o, t_0) = y_3^o W(\bar{t}^o)$$

Then we have

$$P_h(\bar{t}^o, t_0^o) = (\phi(\bar{t}^o), y_3^o)$$

and so $P_h(\mathbb{P}^2)$ is dense in $\text{Cyl}(\mathcal{C})$. □

Now we are ready to introduce the projective auxiliary polynomials. We consider the following system:

$$\mathfrak{S}_4^{P_h}(d, \bar{k}) \equiv \begin{cases} S_0(\bar{k}, \bar{t}_h) := k_1(Y N_3 - Z N_2) - k_2(X N_3 - Z N_1) + k_3(X N_2 - Y N_1) \\ S_1(d, \bar{k}, \bar{t}_h) := H(\bar{t}_h)(k_2 Z - k_3 Y)^2 - d^2 W(\bar{t}_h)^2 (k_2 N_3 - k_3 N_2)^2 \\ S_2(d, \bar{k}, \bar{t}_h) := H(\bar{t}_h)(k_1 Z - k_3 X)^2 - d^2 W(\bar{t}_h)^2 (k_1 N_3 - k_3 N_1)^2 \\ S_3(d, \bar{k}, \bar{t}_h) := H(\bar{t}_h)(k_1 Y - k_2 X)^2 - d^2 W(\bar{t}_h)^2 (k_1 N_2 - k_2 N_1)^2 \end{cases} \quad (22)$$

As usual, for $(d^o, \bar{k}^o) \in \mathbb{C}^4$, we denote by $\Psi_4^{P_h}(d^o, \bar{k}^o)$ the set of projective solutions of $\mathfrak{S}_4^{P_h}(d^o, \bar{k}^o)$. Our next goal is the analysis of the relation between $\Psi_4^{P_h}(d^o, \bar{k}^o)$ and $\Psi_3^P(d^o, \bar{k}^o)$ (the set of solutions of $\mathfrak{S}_3^P(d^o, \bar{k}^o)$, see Subsection 2.2, page 38). In particular, in order to obtain the degree formula, we will characterize those points in $\Psi_4^{P_h}(d^o, \bar{k}^o)$ that correspond to the points in $\mathcal{A} \cap \Psi_3^P(d^o, \bar{k}^o)$. In Proposition 2.18 (page 46) we have seen that the invariant solutions of $\Psi_3^P(d, \bar{k})$ correspond to fake points. Thus, as a first step, we will characterize certain invariant solutions of $\mathfrak{S}_4^{P_h}(d, \bar{k})$.

Lemma 3.4. *Let $S = c_1 S_1 + c_2 S_2 + c_3 S_3$. Then:*

$$\text{Con}_{(d, \bar{k})}(S(\bar{c}, d, \bar{k}, \bar{t}_h)) = \gcd(H, W^2).$$

Proof. Since $S = c_1 S_1 + c_2 S_2 + c_3 S_3$, one has:

$$\text{Con}_{(d, \bar{k})}(S) = \gcd(\text{Con}_{(d, \bar{k})}(S_1), \text{Con}_{(d, \bar{k})}(S_2), \text{Con}_{(d, \bar{k})}(S_3)).$$

Now, considering S_i for $i = 1, 2, 3$ as polynomials in $\mathbb{C}[\bar{t}_h][d, \bar{k}]$ one has:

$$\text{Con}_{(d, \bar{k})}(S_1) = \gcd(HZ^2, HZY, HY^2, W^2 N_2^2, W^2 N_2 N_3, W^2 N_3^2).$$

That is,

$$\text{Con}_{(d, \bar{k})}(S_1) = \gcd(H \gcd(Y, Z)^2, W^2 \gcd(N_2, N_3)).$$

Similarly,

$$\text{Con}_{(d, \bar{k})}(S_2) = \gcd(H \gcd(X, Z)^2, W^2 \gcd(N_1, N_3)).$$

and

$$\text{Con}_{(d, \bar{k})}(S_3) = \gcd(H \gcd(X, Y)^2, W^2 \gcd(N_1, N_2)).$$

Taking into account that $\gcd(N_1, N_2, N_3) = 1$ and $\gcd(X, Y, Z, W) = 1$, one has

$$\text{Con}_{(d, \bar{k})}(S) = \gcd(H \gcd(X, Y, Z)^2, W^2) = \gcd(H, W^2).$$

□

In order to use the above results, and to state the degree formula, we need to introduce some additional notation. We denote by:

$$Q_0(\bar{t}_h) = \text{Con}_{\bar{k}}(S_0(\bar{k}, \bar{t}_h)) \quad \text{and} \quad Q(\bar{t}_h) = \text{Con}_{(d, \bar{k})}(S(\bar{c}, d, \bar{k}, \bar{t}_h)). \quad (23)$$

Observe that, by Lemma 3.4, Q does not depend on \bar{c} , a fact that is reflected in our notation. Furthermore, note that:

$$Q_0(\bar{t}_h) = \gcd(YN_3 - ZN_2, XN_3 - ZN_1, XN_2 - YN_1)$$

and

$$Q(\bar{t}_h) = \gcd(H, W^2).$$

We also denote by:

$$\tilde{H}(\bar{t}_h) = \frac{H(\bar{t}_h)}{Q(\bar{t}_h)}, \quad \tilde{W}(\bar{t}_h) = \frac{W^2(\bar{t}_h)}{Q(\bar{t}_h)}, \quad (24)$$

and

$$\begin{cases} U_1(\bar{t}_h) = \frac{(Y N_3 - Z N_2)(\bar{t}_h)}{Q_0(\bar{t}_h)}, \\ U_2(\bar{t}_h) = \frac{(Z N_1 - X N_3)(\bar{t}_h)}{Q_0(\bar{t}_h)}, \\ U_3(\bar{t}_h) = \frac{(X N_2 - Y N_1)(\bar{t}_h)}{Q_0(\bar{t}_h)}. \end{cases} \quad (25)$$

Thus, one has:

$$S_0(\bar{k}, \bar{t}_h) = Q_0(\bar{t}_h)(k_1 U_1(\bar{t}_h) + k_2 U_2(\bar{t}_h) + k_3 U_3(\bar{t}_h)).$$

We denote as well:

$$\begin{cases} M_{h,1}(\bar{k}, \bar{t}_h) = k_2 Z(\bar{t}_h) - k_3 Y(\bar{t}_h), & G_{h,1}(\bar{k}, \bar{t}_h) = k_2 N_3(\bar{t}_h) - k_3 N_2(\bar{t}_h), \\ M_{h,2}(\bar{k}, \bar{t}_h) = k_3 X(\bar{t}_h) - k_1 Z(\bar{t}_h), & G_{h,2}(\bar{k}, \bar{t}_h) = k_3 N_1(\bar{t}_h) - k_1 N_3(\bar{t}_h), \\ M_{h,3}(\bar{k}, \bar{t}_h) = k_1 Y(\bar{t}_h) - k_2 X(\bar{t}_h), & G_{h,3}(\bar{k}, \bar{t}_h) = k_1 N_2(\bar{t}_h) - k_2 N_1(\bar{t}_h). \end{cases} \quad (26)$$

and so, for $i = 1, 2, 3$,

$$S_i(d, \bar{k}, \bar{t}_h) = Q(\bar{t}_h) \left(\tilde{H}(\bar{t}_h) M_{h,i}^2(\bar{k}, \bar{t}_h) - d^2 \tilde{W}(\bar{t}_h) G_{h,i}^2(\bar{k}, \bar{t}_h) \right)$$

We denote:

$$T_0(\bar{k}, \bar{t}_h) = \frac{S_0(\bar{k}, \bar{t}_h)}{Q_0(\bar{t}_h)} \quad (27)$$

and, for $i = 1, 2, 3$,

$$T_i(d, \bar{k}, \bar{t}_h) = \frac{S_i(d, \bar{k}, \bar{t}_h)}{Q(\bar{t}_h)}, \quad (28)$$

Finally, we denote:

$$T(\bar{c}, d, \bar{k}, \bar{t}_h) = \frac{S(\bar{c}, d, \bar{k}, \bar{t}_h)}{Q(\bar{t}_h)}. \quad (29)$$

Note that:

$$T(\bar{c}, d, \bar{k}, \bar{t}_h) = c_1 T_1(d, \bar{k}, \bar{t}_h) + c_2 T_2(d, \bar{k}, \bar{t}_h) + c_3 T_3(d, \bar{k}, \bar{t}_h).$$

With this notation we can introduce the system of equations that will play the central role in the degree formula:

$$\mathfrak{S}_5^{P_h}(d, \bar{k}) \equiv \begin{cases} T_0(\bar{k}, \bar{t}_h) = k_1 U_1(\bar{t}_h) + k_2 U_2(\bar{t}_h) + k_3 U_3(\bar{t}_h) = 0 \\ T_i(d, \bar{k}, \bar{t}_h) = \tilde{H}(\bar{t}_h) M_{h,i}^2(\bar{k}, \bar{t}_h) - d^2 \tilde{W}(\bar{t}_h) G_{h,i}^2(\bar{k}, \bar{t}_h), \\ \text{for } i = 1, 2, 3. \end{cases} \quad (30)$$

We will refer to this as the **Projective Auxiliary System**.

3.2 Invariant solutions of the projective auxiliary system

In passing from $\mathfrak{S}_3^P(d, \bar{k})$ to $\mathfrak{S}_4^{P_h}(d, \bar{k})$, and then to $\mathfrak{S}_5^{P_h}(d, \bar{k})$, we have introduced additional solutions at infinity, in the space of parameters (that is, with $t_0 = 0$). The following results will show that, in a certain open subset of values of (d, \bar{k}) , these solutions at infinity are invariant w.r.t. (d, \bar{k}) . We start with some technical lemmas.

Lemma 3.5. *There is always $i^o \in \{1, 2, 3\}$ such that $U_{i^o}(0, t_1, t_2)$ and $T_{i^o}(d, \bar{k}, 0, t_1, t_2)$ are both not identically zero.*

Proof. First, let us prove that there are always $i, j \in \{1, 2, 3\}$ such that t_0 does not divide T_i and T_j . Suppose, on the contrary that, for example $T_1(d, \bar{k}, 0, t_1, t_2) \equiv 0$ and $T_2(d, \bar{k}, 0, t_1, t_2) \equiv 0$. Considering T_1 and T_2 as polynomials in $\mathbb{C}[\bar{t}][d, \bar{k}]$, if t_0 divides T_1 and T_2 one concludes that t_0 must divide

$$\tilde{H}X, \tilde{H}Y, \tilde{H}Z, \tilde{W}N_1, \tilde{W}N_2 \text{ and } \tilde{W}N_3.$$

If one assumes that t_0 divides \tilde{W} , then it does not divide \tilde{H} , because $\gcd(\tilde{H}, \tilde{W}) = 1$. Thus it divides X, Y and Z . But this is again a contradiction, since $\gcd(X, Y, Z, W) = 1$, and \tilde{W} divides W . Thus, t_0 does not divide \tilde{W} . Then it must divide N_1, N_2, N_3 . This is also a contradiction, since $\gcd(N_1, N_2, N_3) = 1$. Therefore we can assume w.l.o.g. that e.g. t_0 does not divide T_1 and T_2 . To finish the proof in this case we need to show that, if t_0 divides T_3 , then it does not divide at least one of U_1 and U_2 . The hypothesis that t_0 divides T_3 implies that it divides

$$\tilde{H}X, \tilde{H}Y, \tilde{W}N_1 \text{ and } \tilde{W}N_2.$$

If t_0 divides \tilde{W} , again, it must divide X and Y . Thus it does not divide Z . Now, observe that $XU_1 + YU_2 + ZU_3 = 0$. Therefore, one concludes that t_0 divides U_3 . Thus, t_0 does not divide at least one of U_1 and U_2 , since $\gcd(U_1, U_2, U_3) = 1$. If t_0 does not divide \tilde{W} , then it divides N_1 and N_2 . Observing that $N_1U_1 + N_2U_2 + N_3U_3 = 0$, we again conclude that t_0 does not divide at least one of U_1 and U_2 , since $\gcd(U_1, U_2, U_3) = 1$. \square

Lemma 3.6. *Let $i^o \in \{1, 2, 3\}$ be such that $T_{i^o}(d, \bar{k}, 0, t_1, t_2)$ and $U_{i^o}(0, t_1, t_2)$ are both not identically zero (see Lemma 3.5). Then*

$$\gcd(T_0(\bar{k}, 0, t_1, t_2), T_{i^o}(d, \bar{k}, 0, t_1, t_2))$$

does not depend on \bar{k} (it certainly does not depend on d).

Proof. The claim follows observing that $T_0(\bar{k}, 0, t_1, t_2)$ depends linearly on k_{i^o} , and $T_{i^o}(d, \bar{k}, 0, t_1, t_2)$ does not. \square

In order to describe what we mean when we say that a solution is invariant w.r.t. (d, \bar{k}) , we make the following definition (recall Definition 2.15, page 44):

Definition 3.7. *Let Ω_1 be as in Lemma 2.17 (page 45), and let Ω be a non-empty open subset of Ω_1 . The set of invariant solutions of $\mathfrak{S}_5^{P_h}(d, \bar{k})$ w.r.t. Ω is defined as the set:*

$$\mathcal{I}_5^{P_h}(\Omega) = \bigcap_{(d^o, \bar{k}^o) \in \Omega} \Psi_5^{P_h}(d^o, \bar{k}^o)$$

Remark 3.8.

1. *Considering T_i (for $i = 0, \dots, 3$) as polynomials in $\mathbb{C}[\bar{t}_h][d, \bar{k}]$, it is easy to see that $\mathcal{I}_5^{P_h}(\Omega)$ is the set of solutions of:*

$$\begin{cases} U_1(\bar{t}_h) = U_2(\bar{t}_h) = U_3(\bar{t}_h) = 0, \\ (\tilde{H} \cdot X)(\bar{t}_h) = (\tilde{H} \cdot Y)(\bar{t}_h) = (\tilde{H} \cdot Z)(\bar{t}_h) = 0, \\ (\tilde{W} \cdot N_1)(\bar{t}_h) = (\tilde{W} \cdot N_2)(\bar{t}_h) = (\tilde{W} \cdot N_3)(\bar{t}_h) = 0. \end{cases} \quad (31)$$

2. *In particular, since $\gcd(U_1, U_2, U_3) = 1$, the set $\mathcal{I}_5^{P_h}(\Omega)$ is always a finite set.*

The following proposition shows that, restricting the values of (d, \bar{k}) to a certain open set, we can ensure that all the solutions at infinity of $\mathfrak{S}_5^{P_h}(d, \bar{k})$ are invariant w.r.t. the particular choice of (d, \bar{k}) in that open set.

Proposition 3.9. *There exists an open non-empty subset $\Omega_2 \subset \Omega_1$ (with Ω_1 as in Lemma 2.17, page 45), such that if $(d^o, \bar{k}^o) \in \Omega_2$, and $\bar{t}_h^o = (0 : t_1^o : t_2^o) \in \Psi_5^{P_h}(d^o, \bar{k}^o)$, then $\bar{t}_h^o \in \mathcal{I}_5^{P_h}(\Omega_2)$.*

Proof. We know that $T_0(\bar{k}, 0, \bar{t}) \not\equiv 0$. Suppose, in the first place, that $T_0(\bar{k}, 0, \bar{t})$ depends only in \bar{k} and one of the variables t_1, t_2 . Say, e.g., $T_0(\bar{k}, 0, \bar{t}) = T_0^*(\bar{k})t_1^p$ for some $p \in \mathbb{N}$. This implies that, for any given (d^o, \bar{k}^o) such that $T_0^*(\bar{k}^o) \neq 0$, $(0 : 0 : 1)$ is the only possible point of $\Psi_5^{P_h}(d^o, \bar{k}^o)$ with $t_0 = 0$. Obviously, if

$$T_i(d, \bar{k}, 0, 0, 1) \equiv 0 \text{ for } i = 1, 2, 3,$$

then one may take $\Omega_2 = \Omega_1 \cap \{(d^o, \bar{k}^o) \in \mathbb{C}^4 / T_0^*(\bar{k}^o) \neq 0\}$, and the result is proved. On the other hand, if not all $T_i(d, \bar{k}, 0, 0, 1) \equiv 0$, say w.l.o.g that

$$T_1(d, \bar{k}, 0, 0, 1) \not\equiv 0$$

then we may take $\Omega_2 = \Omega_1 \cap \{(d^o, \bar{k}^o) / T_0^*(\bar{k}^o) T_1(d^o, \bar{k}^o, 0, 0, 1) \neq 0\}$, and the result is proved.

Thus, w.l.o.g. we can assume that $T_0(\bar{k}, 0, \bar{t})$ depends on both t_1 and t_2 . Let $i^o \in \{1, 2, 3\}$ be such that $U_i(0, \bar{t}) \not\equiv 0$ and $T_i(d, \bar{k}, 0, \bar{t}) \not\equiv 0$ (see Lemma 3.6, 53). By Lemma 3.5 (page 52) we know that this is the case at least for one value of i^o . Let us consider (see Lemma 3.6):

$$T_{i^o}^*(\bar{t}) = \gcd(T_0(\bar{k}, 0, \bar{t}), T_{i^o}(d, \bar{k}, 0, \bar{t})) \in \mathbb{C}[\bar{t}].$$

Note that $T_{i^o}^*$ is homogeneous in \bar{t} , and so, if it is not constant, it factors as:

$$T_{i^o}^*(\bar{t}) = \gamma \prod_{j=1}^p (\beta_j t_1 - \alpha_j t_2)$$

for some $\gamma \in \mathbb{C}^\times$, and $(\alpha_j, \beta_j) \in \mathbb{C}^2, j = 1, \dots, p$. For each point $(0 : \alpha_j : \beta_j)$ we can repeat the construction that we did for $(0 : 0 : 1)$. Thus, one obtains a non-empty open set $\Omega_2^1 \subset \Omega_1$ such that, if $(d^o, \bar{k}^o) \in \Omega_2^1$, and $T_{i^o}^*(\alpha_j, \beta_j) = 0$, then either $(0 : \alpha_j : \beta_j) \notin \Psi_5^{P_h}(d^o, \bar{k}^o)$, or $(0 : \alpha_j : \beta_j) \in \mathcal{I}_5^{P_h}(\Omega_2^1)$.

Let

$$T'_0(\bar{k}, \bar{t}) = \frac{T_0(\bar{k}, 0, \bar{t})}{T_{i^o}^*(\bar{t})}, \quad \text{and} \quad T'_{i^o}(d, \bar{k}, \bar{t}) = \frac{T_{i^o}(d, \bar{k}, 0, \bar{t})}{T_{i^o}^*(\bar{t})}.$$

Note that both $T'_0(\bar{k}, \bar{t})$ and $T'_{i^o}(d, \bar{k}, \bar{t})$ are homogeneous in \bar{t} , and by construction they have a trivial gcd. If we define:

$$\Gamma(d, \bar{k}, t_2) = \begin{cases} \text{Res}_{t_1}(T'_0(\bar{k}, \bar{t}), T'_{i^o}(d, \bar{k}, \bar{t})) & \text{if } \deg_{t_1}(T'_0(\bar{k}, \bar{t})) > 0, \\ T'_0(\bar{k}, \bar{t}) & \text{in other case.} \end{cases}$$

Then Γ is not identically zero, and since T'_0 and T'_{i^o} are both homogeneous in \bar{t} , we have a factorization:

$$\Gamma(d, \bar{k}, t_2) = t_2^q \Gamma^*(d, \bar{k})$$

for some $q \in \mathbb{N}$. Note also that, by construction, since $\gcd(T'_0, T'_{i^o}) = 1$, t_2 cannot divide both T'_0 and T'_{i^o} . In particular, since these polynomials are homogeneous in \bar{t} , one concludes that $\bar{t}^o = (1, 0)$ is not a solution of

$$T'_0(\bar{k}, \bar{t}) = T'_{i^o}(d, \bar{k}, \bar{t}) = 0.$$

We define

$$\Omega_2 = \Omega_2^1 \cap \{(d^o, \bar{k}^o)/\Gamma^*(d^o, \bar{k}^o) \neq 0\}$$

If $(d^o, \bar{k}^o) \in \Omega_2$, and $\bar{t}_h^o = (0 : t_1^o : t_2^o) \in \Psi_5^{P_h}(d^o, \bar{k}^o)$, then either $T'_0(\bar{k}^o, \bar{t}^o) \neq 0$ or $T'_{i^o}(d^o, \bar{k}^o, \bar{t}^o) \neq 0$. In any case, one has $T_{i^o}^*(\bar{t}^o) = 0$ (that is, $(0 : t_1^o : t_2^o) = (0 : \alpha_j : \beta_j)$ for some $j = 1, \dots, p$). The construction of Ω_2^1 implies that $(0 : t_1^o : t_2^o) \in \mathcal{I}_5^{P_h}(\Omega_2)$. \square

Let \mathcal{A}_h denote the set of values $\bar{t}_h^o \in \mathbb{P}^2$ such that (compare to the Definition of the set \mathcal{A} in Equation 16, page 40):

$$\left\{ \begin{array}{l} t_0^o W(\bar{t}_h^o) H(\bar{t}_h^o) (Y(\bar{t}_h^o) N_3(\bar{t}_h^o) - Z(\bar{t}_h^o) N_2(\bar{t}_h^o)) \neq 0 \\ \text{or} \\ t_0^o W(\bar{t}_h^o) H(\bar{t}_h^o) (Z(\bar{t}_h^o) N_1(\bar{t}_h^o) - X(\bar{t}_h^o) N_3(\bar{t}_h^o)) \neq 0 \\ \text{or} \\ t_0^o W(\bar{t}_h^o) H(\bar{t}_h^o) (X(\bar{t}_h^o) N_1(\bar{t}_h^o) - Y(\bar{t}_h^o) N_1(\bar{t}_h^o)) \neq 0 \end{array} \right. \quad (32)$$

Equivalently, \mathcal{A}_h consists in those points $\bar{t}_h^o \in \mathbb{P}^2$ such that:

$$t_0^o W(\bar{t}_h^o) H(\bar{t}_h^o) \neq 0 \text{ and } (X(\bar{t}_h^o), Y(\bar{t}_h^o), Z(\bar{t}_h^o)) \wedge \bar{N}(\bar{t}_h^o) \neq \bar{0}. \quad (33)$$

We will see that the non-invariant solutions of $\Psi_5^{P_h}(d, \bar{k})$ are points in \mathcal{A}_h . Note that we are explicitly asking these points to be affine (recall Proposition 3.9, page 53)

With this notation we are ready to state the main theorem about System $\mathfrak{S}_5^{P_h}(d, \bar{k})$.

Theorem 3.10. *Let Ω_2 be as in Proposition 3.9. If $(d^o, \bar{k}^o) \in \Omega_2$,*

$$\bar{t}^o = (t_1^o, t_2^o) \in \mathcal{A} \cap \Psi_3^P(d^o, \bar{k}^o) \Leftrightarrow \bar{t}_h^o = (1 : t_1^o : t_2^o) \in \mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o)$$

Recall that every point in \mathcal{A}_h is affine, see Equation 16 (page 40), for the Definition of \mathcal{A} , and page 40 for the definition of $\Psi_3^P(d^o, \bar{k}^o)$.

Proof. Let us prove that \Rightarrow holds. If $\bar{t}^o \in \mathcal{A} \cap \Psi_3^P(d^o, \bar{k}^o)$, then, e.g.,

$$P_0(\bar{t}^o) h(\bar{t}^o) (P_2 n_3 - P_3 n_2)(\bar{t}^o) \neq 0.$$

Therefore,

$$W(\bar{t}_h^o) H(\bar{t}_h^o) (Y N_3 - Z N_2)(\bar{t}_h^o) \neq 0,$$

and so $\bar{t}_h^o \in \mathcal{A}_h$. Besides, this last inequality implies that $Q_0(\bar{t}_h^o) Q(\bar{t}_h^o) \neq 0$. Since $\bar{t}^o \in \mathcal{A} \cap \Psi_3^P(d^o, \bar{k}^o)$, by Proposition 2.11(b) (page 40), one has that $(1 : t_1^o : t_2^o) \in \Psi_4^{P_h}(d^o, \bar{k}^o)$. From Equations 27 and 28 (page 51), and from $Q_0(\bar{t}_h^o) Q(\bar{t}_h^o) \neq 0$, one concludes that $\bar{t}_h^o = (1 : t_1^o : t_2^o) \in \mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o)$. Thus, \Rightarrow is proved.

The proof of \Leftarrow is similar, simply reversing the implications. \square

Remark 3.11. Let Ω_2 be as in Proposition 3.9, and let $(d^o, \bar{k}^o) \in \Omega_2$. Then, Theorem 3.10 implies that:

$$m\delta = \# \left(\mathcal{A} \cap \Psi_3^P(d^o, \bar{k}^o) \right) = \# \left(\mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o) \right).$$

Theorem 3.10 establishes the link between $\mathcal{A} \cap \Psi_3^P(d^o, \bar{k}^o)$ and $\mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o)$ for a fixed $(d^o, \bar{k}^o) \in \Omega_2$. As we said before, the non-invariant solutions of $\Psi_5^{P_h}(d, \bar{k})$ should be the points in \mathcal{A}_h . As a first step, we have this result.

Proposition 3.12. Let Ω_2 be as in Proposition 3.9 (page 53). If $\bar{t}_h^o \in \mathcal{A}_h$, then $\bar{t}_h^o \notin \mathcal{I}_5^{P_h}(\Omega_2)$.

Proof. Let us suppose that for every $(d^o, \bar{k}^o) \in \Omega_2$, one has $\bar{t}_h^o \in \left(\mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o) \right)$. Then (recall Equation 16, page 40), \bar{t}_h^o is of the form $(1 : t_1^o : t_2^o)$ and, by Theorem 3.10,

$$\bar{t}^o = (t_1^o, t_2^o) \in \bigcap_{(d^o, \bar{k}^o) \in \Omega_2} \Psi_3^P(d^o, \bar{k}^o) = \mathcal{I}_3^P(\Omega_2).$$

Since $\Omega_2 \subset \Omega_1$, Proposition 2.18 (page 46) applies, and one concludes that $(t_1^o, t_2^o) \in \mathcal{F}$. But then,

$$P_0(\bar{t}^o)h(\bar{t}^o) = 0 \text{ or } (P_1(\bar{t}^o), P_2(\bar{t}^o), P_3(\bar{t}^o)) \wedge \bar{n}(\bar{t}^o) = \bar{0}$$

This implies that:

$$W(\bar{t}_h^o)H(\bar{t}_h^o) = 0 \text{ or } (X(\bar{t}_h^o), Y(\bar{t}_h^o), Z(\bar{t}_h^o)) \wedge \bar{N}(\bar{t}_h^o) = \bar{0},$$

contradicting $\bar{t}_h^o \in \mathcal{A}_h$. □

As we said before, we will prove the converse of this proposition. That is, if $\bar{t}_h^o \in \Psi_5^{P_h}(d^o, \bar{k}^o)$ for some $(d^o, \bar{k}^o) \in \Omega_2$, but $\bar{t}_h^o \notin \mathcal{I}_5^{P_h}(\Omega_2)$, then we would like to conclude that $\bar{t}_h^o \in \mathcal{A}_h$. However, the open set Ω_2 can be too large for this to hold. More precisely, the problem is caused by the solutions of $Q(\bar{t}_h)Q_0(\bar{t}_h) = 0$ (when Q and Q_0 are not equal 1). These points do not belong to \mathcal{A}_h . However, since $\mathcal{I}_5^{P_h}(\Omega_2)$ is finite (see Remark 3.8, page 53), most of the solutions $Q(\bar{t}_h)Q_0(\bar{t}_h) = 0$ are not invariant. Therefore, we need to impose some more restrictions in the values of (d, \bar{k}) . Note, however, that we have already dealt with the points at infinity; thus, we need only consider the affine solutions of $Q(\bar{t}_h)Q_0(\bar{t}_h) = 0$. We do the necessary technical work in the following lemma. First, we introduce some notation for the affine versions of some polynomials. We denote:

$$\begin{cases} q(\bar{t}) = Q(1, t_1, t_2), q_0(\bar{t}) = Q_0(1, t_1, t_2), \tilde{w}(\bar{t}) = \tilde{W}(1, t_1, t_2), \tilde{h}(\bar{t}) = \tilde{H}(1, t_1, t_2), \\ u_i(\bar{t}) = U_i(1, t_1, t_2) \text{ for } i = 1, 2, 3. \end{cases}$$

and we consider a new auxiliary set of variables $\bar{\rho} = (\rho_1, \rho_2, \rho_3)$, in order to do the necessary Rabinowitsch's tricks.

Let $\mathcal{G} \subset \mathbb{C} \times \mathbb{C}^3 \times \mathbb{C}^2 \times \mathbb{C}^2$ be the set of solutions (in the variables $(d, \bar{k}, \bar{\rho}, \bar{t})$) of this system of equations:

$$\begin{cases} q(\bar{t})q_0(\bar{t}) = 0 \\ \tilde{h}(\bar{t})M_i^2(\bar{k}, \bar{t}) - d^2\tilde{w}(\bar{t})G_i^2(\bar{k}, \bar{t}) = 0 & \text{for } i = 1, 2, 3. \\ k_1u_1(\bar{t}) + k_2u_2(\bar{t}) + k_3u_3(\bar{t}) = 0 \\ \rho_1\tilde{w}(\bar{t})\tilde{h}(\bar{t}) - 1 = 0 \\ \prod_{i=1}^3(\rho_2u_i(\bar{t}) - 1) = 0 \\ \prod_{i=1}^3(\rho_3n_i(\bar{t}) - 1) = 0 \end{cases} \quad (34)$$

and consider the projection $\pi_1(d, \bar{k}, \lambda, \bar{\rho}, \bar{t}) = (d, \bar{k})$.

Lemma 3.13. *\mathcal{G} is empty or $\dim(\pi_1(\mathcal{G})) \leq 3$.*

Proof. Let $\mathcal{G} \neq \emptyset$. Then $q(\bar{t})q_0(\bar{t})$ is not constant. We will use Lemma 1.4 (page 12), to prove that $\dim(\mathcal{G}) \leq 3$. From this the result follows immediately. Consider the projection $\pi_2(d, \bar{k}, \bar{\rho}, \bar{t}) = \bar{t}$. Clearly, $\pi_2(\mathcal{G})$ is contained in the affine curve defined by $q(\bar{t})q_0(\bar{t}) = 0$. Thus, $\dim(\pi_2(\mathcal{G})) \leq 1$. Let $\bar{t}^o \in \pi_2(\mathcal{G})$. First, let us suppose that for all $(d^o, \bar{k}^o, \bar{\rho}^o, \bar{t}^o) \in \pi_2^{-1}(\bar{t}^o)$, one has $\bar{k}^o = \bar{0}$. Then, if $(d^o, \bar{0}, \bar{\rho}^o, \bar{t}^o) \in \pi_2^{-1}(\bar{t}^o)$, ρ_1^o, ρ_2^o , and ρ_3^o must be one of the finitely many solutions of the polynomial equations:

$$\rho_1\tilde{w}(\bar{t}^o)\tilde{h}(\bar{t}^o) - 1 = 0, \quad \prod_{i=1}^3(\rho_2u_i(\bar{t}^o) - 1) = 0, \quad \text{and} \quad \prod_{i=1}^3(\rho_3n_i(\bar{t}^o) - 1) = 0.$$

The condition $\bar{t}^o \in \pi_2(\mathcal{G})$ implies that these equations can be solved. Note that, in this case, $(d^o, \bar{0}, \bar{\rho}^o, \bar{t}^o) \in \pi_2^{-1}(\bar{t}^o)$ does not impose any condition on d^o . It follows that, in this case, one has $\mu = \dim(\pi_2^{-1}(\bar{t}^o)) = 1$.

Now, let us suppose that $(d^o, \bar{k}^o, \bar{\rho}^o, \bar{t}^o) \in \pi_2^{-1}(\bar{t}^o)$, with $\bar{k}^o \neq \bar{0}$. Then, by a similar argument to the proof of Proposition 2.11(a) (page 40), and taking $\tilde{w}(\bar{t}^o)\tilde{h}(\bar{t}^o) \neq 0$ into account, there exists $\lambda^o \in \mathbb{C}^\times$ such that

$$M_i(\bar{k}^o, \bar{t}^o) = \lambda^o G_i(\bar{k}^o, \bar{t}^o) \text{ for } i = 1, 2, 3.$$

Thus, in this case d^o must be a solution of:

$$\tilde{h}(\bar{t}^o)(\lambda^o)^2 - (d^o)^2\tilde{w}(\bar{t}^o) = 0.$$

Besides, there exists also $j^o \in \{1, 2, 3\}$ with $u_{j^o}(\bar{t}^o) \neq 0$. Then, \bar{k}^o must belong to the two-dimensional space defined by

$$k_1u_1(\bar{t}^o) + k_2u_2(\bar{t}^o) + k_3u_3(\bar{t}^o) = 0.$$

Finally, ρ_1^o, ρ_2^o , and ρ_3^o must be one of the finitely many solutions of the polynomial equations:

$$\rho_1 \tilde{w}(\bar{t}^o) \tilde{h}(\bar{t}^o) - 1 = 0, \quad \prod_{i=1}^3 (\rho_2 u_i(\bar{t}^o) - 1) = 0, \quad \text{and} \quad \prod_{i=1}^3 (\rho_3 n_i(\bar{t}^o) - 1) = 0.$$

The condition $\bar{t}^o \in \pi_2(\mathcal{G})$ implies that these equations can be solved. These remarks show that for every $\bar{t}^o \in \pi_2(\mathcal{G})$, one has $\mu = \dim(\pi_2^{-1}(\bar{t}^o)) \leq 2$. Thus, using Lemma 1.4:

$$\dim(\mathcal{G}) = \dim(\pi_2(\mathcal{G})) + \mu \leq 1 + 2 = 3,$$

and the lemma is proved. \square

If \bar{t}_h^o is such that

$$T_0(\bar{k}, \bar{t}_h^o) \equiv 0 \text{ and } T_i(d, \bar{k}, \bar{t}_h^o) \equiv 0, \text{ for } i = 1, 2, 3,$$

then $\bar{t}_h^o \in \mathcal{I}_5^{P_h}(\Omega)$ for any choice of Ω . However, if this is not the case, then sometimes we need to remove from Ω precisely those values (d^o, \bar{k}^o) such that $\bar{t}_h^o \in \Psi_5^{P_h}(d^o, \bar{k}^o)$. In the proof of the following lemma we will need to do this several times. Thus we introduce the necessary notation.

Definition 3.14. Let $\Omega \subset \mathbb{C} \times \mathbb{C}^3$ be non-empty and open. For $\bar{t}_h^o \in \mathbb{P}^2$ we define:

$$\Omega^{inv}(\bar{t}_h^o) = \begin{cases} \Omega, & \text{if } T_0(\bar{k}, \bar{t}_h^o) \equiv 0 \text{ and } T_i(d, \bar{k}, \bar{t}_h^o) \equiv 0, \text{ for } i = 1, 2, 3. \\ \Omega \setminus \{(d^o, \bar{k}^o)/\bar{t}_h^o \in \Psi_5^{P_h}(d^o, \bar{k}^o)\}, & \text{in other case.} \end{cases}$$

Remark 3.15.

(1) Note that if $\Omega \subset \Omega^{inv}(\bar{t}_h^o)$, then $\bar{t}_h^o \in \mathcal{I}_5^{P_h}(\Omega)$.

(2) Observe that $\Omega^{inv}(\bar{t}_h^o) \neq \emptyset$.

Lemma 3.16. Let Ω_2 be as in Proposition 3.9. There exists an open non-empty set $\Omega_3 \subset \Omega_2$ such that the following hold:

(a) If $\bar{t}_h^o \in \Psi_5^{P_h}(d^o, \bar{k}^o)$ for some $(d^o, \bar{k}^o) \in \Omega_3$, and $Q_0(\bar{t}_h^o)Q(\bar{t}_h^o) = 0$, then $\bar{t}_h^o \in \mathcal{I}_5^{P_h}(\Omega_3)$.

(b) If \bar{t}_h^o satisfies

$$T_1(d, k, \bar{t}_h^o) = T_2(d, k, \bar{t}_h^o) = T_3(d, k, \bar{t}_h^o) = 0 \text{ identically in } (d, \bar{k}),$$

then $\bar{t}_h^o \in \mathcal{I}_5^{P_h}(\Omega_3)$.

Proof. Let

$$A_0 = \{\bar{t}_h^o \mid X(\bar{t}_h^o) = Y(\bar{t}_h^o) = Z(\bar{t}_h^o) = W(\bar{t}_h^o) = 0\}.$$

Since $\gcd(X, Y, Z, W) = 1$, one sees that A_0 is (empty or) a finite set. Thus, if we define (see Definition 3.14):

$$\Omega_3^0 = \Omega_2 \cap \left(\bigcap_{\bar{t}_h^o \in A_0} \Omega^{inv}(\bar{t}_h^o) \right).$$

By Remark 3.15, Ω_3^0 is an open non-empty set. Let

$$A_1 = \{\bar{t}_h^o \mid N(\bar{t}_h^o) = \bar{0}\},$$

where $N = (N_1, N_2, N_3)$. Recalling that $\gcd(N_1, N_2, N_3) = 1$, A_1 is (empty or) a finite set. We define:

$$\Omega_3^1 = \Omega_3^0 \cap \left(\bigcap_{\bar{t}_h^o \in A_1} \Omega^{inv}(\bar{t}_h^o) \right).$$

By Remark 3.15, Ω_3^1 is an open non-empty set.

Similarly, since $\gcd(\tilde{H}, \tilde{W}) = 1$, the set

$$A_2 = \{\bar{t}_h^o \mid \tilde{H}(\bar{t}_h^o) = \tilde{W}(\bar{t}_h^o) = 0\}$$

is (empty or) finite. We define

$$\Omega_3^2 = \Omega_3^1 \cap \left(\bigcap_{\bar{t}_h^o \in A_2} \Omega^{inv}(\bar{t}_h^o) \right),$$

and Ω_3^2 is an open non-empty set. Moreover, since $\gcd(U_1, U_2, U_3) = 1$, the set

$$A_3 = \{\bar{t}_h^o \mid U_1(\bar{t}_h^o) = U_2(\bar{t}_h^o) = U_3(\bar{t}_h^o) = 0\}$$

is (empty or) finite. We define

$$\Omega_3^3 = \Omega_3^2 \cap \left(\bigcap_{\bar{t}_h^o \in A_3} \Omega^{inv}(\bar{t}_h^o) \right),$$

and Ω_3^3 is an open non-empty set. We define

$$\Omega_3^4 = \Omega_3^3 \setminus (\pi_1(\mathcal{G})^*),$$

where $\pi_1(\mathcal{G})$ is as in Lemma 3.13 (page 57), and, as usual, the asterisk denotes Zariski closure.

Finally, since $T(\bar{c}, d, \bar{k}, \bar{t})$ is primitive w.r.t. (d, \bar{k}) (recall Equations 23, page 50, and 29, page 51), it follows that the set

$$A_4 = \{\bar{t}_h^o \mid T_1(d, k, \bar{t}_h^o) = T_2(d, k, \bar{t}_h^o) = T_3(d, k, \bar{t}_h^o) = 0 \text{ identically in } (d, \bar{k})\}$$

is (empty or) finite. We define

$$\Omega_3 = \Omega_3^4 \cap \left(\bigcap_{\bar{t}_h^o \in A_4} \Omega^{inv}(\bar{t}_h^o) \right).$$

Let us now suppose that $(d^o, \bar{k}^o) \in \Omega_3$ and $\bar{t}_h^o \in \Psi_5^{P_h}(d^o, \bar{k}^o)$, with $Q_0(\bar{t}_h^o)Q(\bar{t}_h^o) = 0$. We will show that $\bar{t}_h^o \in \mathcal{I}_5^{P_h}(\Omega_3)$. This will prove that statement (a) holds. If \bar{t}_h^o is of the form $(0 : t_1^o : t_2^o)$, by Proposition 3.9, $\bar{t}_h^o \in \mathcal{I}_5^{P_h}(\Omega_3)$ holds trivially. Thus, in the rest of the proof we can assume w.l.o.g. that \bar{t}_h^o is of the form $(1 : t_1^o : t_2^o)$.

If $\bar{t}_h^o \in \cup_{i=0,\dots,3} A_i$, then we have $\Omega_3 \subset \Omega_3^4 \subset \Omega^{inv}(\bar{t}_h^o)$, and by Remark 3.15, $\bar{t}_h^o \in \mathcal{I}_5^{P_h}(\Omega_3)$. So, let $\bar{t}_h^o \notin \cup_{i=0,\dots,3} A_i$. Then the following hold:

- (0) Since $\bar{t}_h^o \notin A_0$, $P_i(\bar{t}^o) \neq 0$ for some $i = 0, \dots, 3$ (recall that $\bar{t}_h^o = (1 : \bar{t}^o)$).
- (1) Since $\bar{t}_h^o \notin A_1$, $N(\bar{t}_h^o) \neq \bar{0}$.
- (2) Since $\bar{t}_h^o \notin A_2$, $\tilde{H}(\bar{t}_h^o) \neq 0$ or $\tilde{W}(\bar{t}_h^o) \neq 0$.
- (3) Since $\bar{t}_h^o \notin A_3$, $U_i(\bar{t}_h^o) \neq 0$ for some $i = 1, 2, 3$.

Let us show that (0) and (2) imply the following:

- (4) $\tilde{H}(\bar{t}_h^o)\tilde{W}(\bar{t}_h^o) \neq 0$.

Indeed, if we suppose that $\tilde{H}(\bar{t}_h^o) = 0$ but $\tilde{W}(\bar{t}_h^o) \neq 0$, then from $\bar{t}_h^o \in \Psi_5^{P_h}(d^o, \bar{k}^o)$ one concludes that $d^o G_i(\bar{k}^o, \bar{t}_h^o) = 0$ for $i = 1, 2, 3$. Since $d^o \neq 0$ and $\bar{k}^o \neq \bar{0}$ in Ω_3 , one has that $N(\bar{t}_h^o)$ is isotropic and parallel to \bar{k}^o , contradicting Lemma 2.17(1) (page 45). On the other hand, if we suppose $\tilde{H}(\bar{t}_h^o) \neq 0$ but $\tilde{W}(\bar{t}_h^o) = 0$, then from $\bar{t}_h^o \in \Psi_5^{P_h}(d^o, \bar{k}^o)$ one concludes that $d^o G_i(\bar{k}^o, \bar{t}_h^o) = 0$ for $i = 1, 2, 3$. Since $d^o \neq 0$, we conclude that $G_i(\bar{k}^o, \bar{t}_h^o) = 0$ for $i = 1, 2, 3$. Thus, \bar{t}^o is a solution of:

$$P_0(\bar{t}) = M_1(\bar{k}^o, 1, \bar{t}) = M_2(\bar{k}^o, 1, \bar{t}) = M_3(\bar{k}^o, 1, \bar{t}) = 0.$$

However, by (0), there exists $j^o \in \{0, 1, 2, 3\}$ such that $P_{j^o}(\bar{t}^o) \neq 0$. Therefore, we get a contradiction with Lemma 2.17(3) (page 45).

From (1), (3), (4), and since $\bar{t}_h^o \in \Psi_5^{P_h}(d^o, \bar{k}^o)$ and $Q_0(\bar{t}_h^o)Q(\bar{t}_h^o) = 0$, it follows that $(d^o, \bar{k}^o, \bar{t}_h^o)$ can be extended to $(d^o, \bar{k}^o, \bar{\rho}^o, \bar{t}^o) \in \mathcal{G}$. Thus, one has $(d^o, \bar{k}^o) \in \pi_1(\mathcal{G})$, contradicting the construction of Ω_3^3 . This finishes the proof of statement (a).

The proof of statement (b) is a consequence of the construction of Ω_3 (in particular, see the construction of A_4); indeed, if \bar{t}_h^o satisfies

$$T_1(d, k, \bar{t}_h^o) = T_2(d, k, \bar{t}_h^o) = T_3(d, k, \bar{t}_h^o) = 0 \text{ identically in } (d, \bar{k}),$$

then $\bar{t}_h^o \in A_4$. It follows that $\Omega_3 \subset \Omega^{inv}(\bar{t}_h^o)$ and so $\bar{t}_h^o \in \mathcal{I}_5^{P_h}(\Omega)$ (see Remark 3.15(1), page 58). \square

Now, restricting the values of (d, \bar{k}) to a new open set, we are ready to prove the announced converse of Proposition 3.12 (page 56).

Proposition 3.17. *Let Ω_3 be as in Lemma 3.16 (page 58). If $\bar{t}_h^o \in \Psi_5^{P_h}(d^o, \bar{k}^o)$ for some $(d^o, \bar{k}^o) \in \Omega_3$, but $\bar{t}_h^o \notin \mathcal{I}_5^{P_h}(\Omega_3)$, then $\bar{t}_h^o \in \mathcal{A}_h$.*

Proof. If $\bar{t}_h^o \notin \mathcal{I}_5^{P_h}(\Omega_3)$, then $t_0^o \neq 0$ (by Proposition 3.9, page 53). Let us write $\bar{t}_h^o = (1 : t_1^o : t_2^o)$. Then, since $\bar{t}_h^o \in \Psi_5^{P_h}(d^o, \bar{k}^o)$, one has $\bar{t}^o \in \Psi_3^P(d^o, \bar{k}^o)$. Note also that, since $\bar{t}_h^o \notin \mathcal{I}_5^{P_h}(\Omega_3)$, by Lemma 3.16, we must have $Q_0(\bar{t}_h^o)Q(\bar{t}_h^o) \neq 0$. If we suppose $\bar{t}_h^o \notin \mathcal{A}_h$, then $\bar{t}^o \notin \mathcal{A}$. Thus $\bar{t}^o \in \mathcal{F}$, and by Proposition 2.18 (page 46), $\bar{t}^o \in \mathcal{I}_3^P(\Omega_3)$. Taking Equation 28 (page 51) into account, and using $Q(\bar{t}_h^o) \neq 0$, we conclude that

$$T_1(d, k, \bar{t}_h^o) = T_2(d, k, \bar{t}_h^o) = T_3(d, k, \bar{t}_h^o) = 0 \text{ identically in } (d, \bar{k}).$$

Then, by Lemma 3.16(b) (page 58), one has that $\bar{t}_h^o \in \mathcal{I}_5^{P_h}(\Omega_3)$. This is a contradiction, and so we obtain that $\bar{t}_h^o \in \mathcal{A}_h$. \square

3.3 Multiplicity of intersection at non-fake points

The auxiliary polynomials S_i (for $i = 0, \dots, 3$) were introduced in Section 2.2 (page 38), in order to reduce the offset degree problem to a problem of intersection between planar curves. More precisely, the preceding results in this paper indicate that the offset degree problem can be reduced to an intersection problem between the planar curves defined by the auxiliary polynomials T_i . A crucial step in this reduction concerns the multiplicity of intersection of these curves at their non-invariant points of intersection. In this subsection we will prove that the value of that multiplicity of intersection is one (in Proposition 3.20, page 62). We first introduce some notation for the curves involved in this problem.

Definition 3.18. *Let Ω_0 be as in Theorem 2.8 (page 35). If $(d^o, \bar{k}^o) \in \Omega_0$, and T_i (for $i = 0, \dots, 3$) are the polynomials introduced in Equations 27 and 28 (page 51), we denote by $\mathcal{T}_0^a(\bar{k}^o)$ (resp. $\mathcal{T}_i^a(d^o, \bar{k}^o)$ for $i = 1, 2, 3$) the affine algebraic set defined by the polynomial $T_0(\bar{k}^o, 1, \bar{t})$ (resp. $T_i(d^o, \bar{k}^o, 1, \bar{t})$ for $i = 1, 2, 3$). Similarly, we denote by $\mathcal{T}_0^h(\bar{k}^o)$ (resp. $\mathcal{T}_i^h(d^o, \bar{k}^o)$ for $i = 1, 2, 3$) the projective algebraic set defined by the polynomial $T_0(\bar{k}^o, \bar{t}_h)$ (resp. $T_i(d^o, \bar{k}^o, \bar{t}_h)$ for $i = 1, 2, 3$).*

Remark 3.19. Note that the homogenization of the polynomials $T_0(\bar{k}^o, 1, \bar{t})$ and $T_i(d^o, \bar{k}^o, 1, \bar{t})$ w.r.t. t_0 does not necessarily coincide with $T_0(\bar{k}^o, \bar{t}_h)$ and $T_i(d^o, \bar{k}^o, \bar{t}_h)$. They may differ in a power of t_0 . In particular, it is not necessarily true that $\overline{\mathcal{T}_0^a(\bar{k}^o)} = \mathcal{T}_0^h(\bar{k}^o)$ and $\overline{\mathcal{T}_i^a(d^o, \bar{k}^o)} = \mathcal{T}_i^h(d^o, \bar{k}^o)$ (the overline denotes projective closure, as usual). However, it holds that $\mathcal{T}_i^h(d^o, \bar{k}^o) \cap \mathbb{C}^n = \mathcal{T}_i^a(d^o, \bar{k}^o)$ and $\mathcal{T}_0^h(d^o, \bar{k}^o) \cap \mathbb{C}^n = \mathcal{T}_0^a(\bar{k}^o)$.

Proposition 3.20. Let Ω_3 be as in Lemma 3.16 (page 58). There exists a non-empty open $\Omega_4 \subset \Omega_3$, such that if $(d^o, \bar{k}^o) \in \Omega_4$, and $\bar{t}_h^o \in \mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o)$, then:

$$\min_{i=1,2,3} (\text{mult}_{\bar{t}^o}(\mathcal{T}_o(\bar{k}^o), \mathcal{T}_i(d^o, \bar{k}^o))) = 1.$$

Proof. Since $\bar{t}_h^o \in \mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o)$, we can write $\bar{t}_h^o = (1 : t_1^o : t_2^o)$. Let $\bar{t}^o = (t_1^o, t_2^o)$. By Theorem 3.10 (page 55) we know that $\bar{t}^o = (t_1^o, t_2^o) \in \mathcal{A} \cap \Psi_3^P(d^o, \bar{k}^o)$. W.l.o.g. we will suppose that

$$P_0(\bar{t}^o)h(\bar{t}^o)(P_2(\bar{t}^o)n_3(\bar{t}^o) - P_3(\bar{t}^o)n_2(\bar{t}^o)) \neq 0$$

(see the definition of the set \mathcal{A} in Equation 16, page 40). In this case, it holds (see Remark 2.12, page 43) that

$$k_2^o n_3(\bar{t}^o) - k_3^o n_2(\bar{t}^o) \neq 0 \text{ and } k_2^o P_3(\bar{t}^o) - k_3^o P_2(\bar{t}^o) \neq 0.$$

Furthermore, by Remark 1.27 (page 24), one has

$$f_j(P(\bar{t})) = \frac{\beta(\bar{t})}{P_0^\mu(\bar{t})} n_j(\bar{t}) \text{ for } j = 1, 2, 3, \quad (35)$$

and therefore

$$\sqrt{h_{\text{imp}}(P(\bar{t}))} = \frac{\beta(\bar{t})}{P_0^\mu(\bar{t})} \sqrt{h(\bar{t})}. \quad (36)$$

with $\beta(\bar{t}^o) \neq 0$ (see Lemma 1.30, page 25).

For this case we will construct a non-empty open set $\Omega_{4,1} \subset \Omega_3$ such that if $(d^o, \bar{k}^o) \in \Omega_{4,1}$, and $\bar{t}_h^o \in \mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o)$, then:

$$\text{mult}_{\bar{t}^o}(\mathcal{T}_o(\bar{k}^o), \mathcal{T}_1(d^o, \bar{k}^o)) = 1.$$

If the second, respectively third, defining equation of \mathcal{A} is used, then analogous open subsets $\Omega_{4,2}$, respectively $\Omega_{4,3}$ can be constructed, and the corresponding result for $\mathcal{T}_2(d^o, \bar{k}^o)$, respectively $\mathcal{T}_3(d^o, \bar{k}^o)$, is obtained. Finally, it suffices to take

$$\Omega_4 = \Omega_{4,1} \cap \Omega_{4,2} \cap \Omega_{4,3}.$$

The construction of $\Omega_{4,1}$ will proceed in several steps:

- (1) By Proposition 2.11 (page 40), $(d^o, \bar{k}^o, \bar{t}^o) \in \pi_{(2,1)}(\Psi_2^P(d^o, \bar{k}^o))$. Thus, by Theorem 2.8 (page 35), the point $\bar{y}^o = P(\bar{t}^o)$ is an affine, non normal-isotropic point of Σ , and it is associated with $\bar{x}^o \in \mathcal{L}_{\bar{k}^o} \cap \mathcal{O}_{d^o}(\Sigma)$, where \bar{x}^o is a non normal-isotropic point of $\mathcal{O}_{d^o}(\Sigma)$. Besides, since $(d^o, \bar{k}^o) \in \Omega_3 \subset \Omega_0$, (see Remark 2.9, page 38), $g(d^o, \bar{x})$ is the defining polynomial of $\mathcal{O}_{d^o}(\Sigma)$.

It follows that there is an open neighborhood U^0 of (d^o, \bar{y}^o) (in the usual unitary topology of $\mathbb{C} \times \mathbb{C}^3$) such that the equation

$$\left(\frac{\partial f}{\partial y_1}(\bar{y})\right)^2 + \left(\frac{\partial f}{\partial y_2}(\bar{y})\right)^2 + \left(\frac{\partial f}{\partial y_3}(\bar{y})\right)^2 = 0$$

has no solutions in U^0 . Similarly, there is an open neighborhood V^0 of (d^o, \bar{x}^o) (in the usual unitary topology of $\mathbb{C} \times \mathbb{C}^3$) such that the equation

$$\left(\frac{\partial g}{\partial x_1}(d, \bar{x})\right)^2 + \left(\frac{\partial g}{\partial x_2}(d, \bar{x})\right)^2 + \left(\frac{\partial g}{\partial x_3}(d, \bar{x})\right)^2 = 0$$

has no solutions in V^0 . Let us consider the map:

$$\varphi : U^0 \rightarrow \mathbb{C}^3$$

defined by

$$\bar{\varphi}(d, \bar{y}) = (\varphi_1(d, \bar{y}), \varphi_2(d, \bar{y}), \varphi_3(d, \bar{y})) = \bar{y} \pm d \frac{\nabla f(\bar{y})}{\sqrt{h_{\text{imp}}(\bar{y})}} \quad (37)$$

We assume w.l.o.g. that the $+$ sign in this expression is chosen so that $\bar{\varphi}(d^o, \bar{y}^o) = \bar{x}^o$; our discussion does not depend on this choice of sign in this expression, as will be shown below. According to Remark 2.12 and Lemma 2.13 (page 43), this implies that:

$$\bar{M}(\bar{k}^o, \bar{t}^o) = \epsilon \frac{d^o P_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}} \bar{G}(\bar{k}^o, \bar{t}^o) \text{ for } i = 1, 2, 3. \quad (38)$$

We will use Equation 38 later in the proof. Since \bar{y}^o is not normal-isotropic in Σ , it follows that $\bar{\varphi}$ is analytic in U^0 . Furthermore, we consider the map

$$\bar{\eta} : V^0 \rightarrow \mathbb{C}^3$$

defined by:

$$\bar{\eta}(d, \bar{x}) = \bar{x} + d \frac{\nabla_{\bar{x}} g(d, \bar{x})}{\|\nabla_{\bar{x}} g(d, \bar{x})\|}.$$

Here $\nabla_{\bar{x}}$ refers to the gradient computed w.r.t. \bar{x} ; that is:

$$\nabla_{\bar{x}} g(d, \bar{x}) = \left(\frac{\partial g}{\partial x_1}(d, \bar{x}), \frac{\partial g}{\partial x_2}(d, \bar{x}), \frac{\partial g}{\partial x_3}(d, \bar{x}) \right).$$

In the definition of $\bar{\eta}$, w.l.o.g. the sign $+$ is chosen so that $\bar{\eta}(d^o, \bar{x}^o) = \bar{y}^o$. Then, since \bar{x}^o is non normal-isotropic in $\mathcal{O}_{d^o}(\Sigma)$, it follows that $\bar{\eta}$ is analytic in V^o . Thus, there are open neighborhoods U^1 of (d^o, \bar{y}^o) and V^1 of (d^o, \bar{x}^o) (in the unitary topology of $\mathbb{C} \times \mathbb{C}^3$), such that $\bar{\varphi}$ is an analytic isomorphism between U^1 and V^1 , with inverse given by $\bar{\eta}$. We can assume w.l.o.g. that $\|\nabla f(\bar{y})\| \neq 0$ holds in U^1 , and $\|\nabla_{d, \bar{x}} g(\bar{x})\| \neq 0$ holds in V^1 . Note also that if $(d^o, \bar{y}^1) \in U^1$, with $\bar{y}^1 \in \Sigma$, then $\bar{\varphi}(d^o, \bar{y}^1) \in \mathcal{O}_{d^o}(\Sigma)$. It follows that the map $\bar{\varphi}_{d^o}$, obtained by restricting $\bar{\varphi}$ to $d = d^o$, induces an isomorphism:

$$d\bar{\varphi}_{d^o} : T_{\bar{y}^o}(\Sigma) \rightarrow T_{\bar{x}^o}(\mathcal{O}_{d^o}(\Sigma))$$

where $T_{\bar{y}^o}(\Sigma)$ is the tangent plane to Σ at \bar{y}^o , and $T_{\bar{x}^o}(\mathcal{O}_{d^o}(\Sigma))$ is the tangent plane to $\mathcal{O}_{d^o}(\Sigma)$ at \bar{x}^o .

Since $(d^o, \bar{k}^o) \in \Omega_3 \subset \Omega_0$, we have $\bar{t}^o \in \Upsilon_1$, with Υ_1 as in Lemma 1.24, page 23 (see the construction of Ω_0^4 in the proof of Theorem 2.8, 35). Thus, the jacobian $\frac{\partial P}{\partial \bar{t}}(\bar{t}^o)$ has rank two. It follows that P induces an isomorphism:

$$dP : T_{\bar{t}^o}(\mathbb{C}^2) \rightarrow T_{\bar{y}^o}(\Sigma)$$

where $T_{\bar{t}^o}(\mathbb{C}^2)$ is the tangent plane to \mathbb{C}^2 at \bar{t}^o . Therefore, the map defined by

$$\bar{\nu}_{d^o}(\bar{t}) = \bar{\varphi}_{d^o}(P(\bar{t})) = \bar{\varphi}(d^o, P(\bar{t})) \quad (39)$$

induces an isomorphism $d\bar{\nu}_{d^o}$ between $T_{\bar{t}^o}(\mathbb{C}^2)$ and $T_{\bar{x}^o}(\mathcal{O}_{d^o}(\Sigma))$.

(2) Consider the following polynomials in $\mathbb{C}[d, \bar{k}, \rho]$:

$$\begin{cases} K(d, \bar{k}, \rho) = k_1 \frac{\partial g}{\partial x_1}(d, \rho \bar{k}) + k_2 \frac{\partial g}{\partial x_2}(d, \rho \bar{k}) + k_3 \frac{\partial g}{\partial x_3}(d, \rho \bar{k}) \\ \tilde{g}(d, \bar{k}, \rho) = g(d, \rho \bar{k}) \end{cases}$$

and let

$$\Theta(d, \bar{k}) = \text{Res}_\rho(K(d, \bar{k}, \rho), \tilde{g}(d, \bar{k}, \rho)).$$

Let us show that this resultant does not vanish identically. If it does, then there are $A, B_1, B_2 \in \mathbb{C}[d, \bar{k}, \rho]$, with $\deg_\rho(A(d, \bar{k}, \rho)) > 0$, such that

$$\begin{cases} K(d, \bar{k}, \rho) = A(d, \bar{k}, \rho)B_1(d, \bar{k}, \rho), \\ \tilde{g}(d, \bar{k}, \rho) = A(d, \bar{k}, \rho)B_2(d, \bar{k}, \rho). \end{cases}$$

Then $g(d, \rho \bar{k}) = A(d, \bar{k}, \rho)B_2(d, \bar{k}, \rho)$, and $\deg_{\bar{k}}(A(d, \bar{k}, \rho)) > 0$ (because \tilde{g} cannot have a non constant factor in $\mathbb{C}[d, \rho]$). Thus, setting $\rho = 1$ and $\bar{k} = \bar{x}$, one has $g(d, \bar{x}) = A(d, \bar{x}, 1)B_2(d, \bar{x}, 1)$. It follows (see Remark 1.18(1), page 19) that if

$\tilde{A}(d, \bar{x})$ is any irreducible factor of $A(d, \bar{x}, 1)$, then $\tilde{A}(d, \bar{x})$ defines an irreducible component \mathcal{M} of the generic offset, such that

$$x_1 \frac{\partial g}{\partial x_1}(d, \bar{x}) + x_2 \frac{\partial g}{\partial x_2}(d, \bar{x}) + x_3 \frac{\partial g}{\partial x_3}(d, \bar{x}) = 0$$

holds identically on \mathcal{M} . Besides, for an open set of points $\bar{x}^o \in \mathcal{M}$, one has $\nabla_{\bar{x}} g(d, \bar{x}^o) = \nabla_{\bar{x}} \tilde{A}(d, \bar{x}^o)$. Thus the above equation implies that

$$x_1 \frac{\partial \tilde{A}}{\partial x_1}(d, \bar{x}) + x_2 \frac{\partial \tilde{A}}{\partial x_2}(d, \bar{x}) + x_3 \frac{\partial \tilde{A}}{\partial x_3}(d, \bar{x}) = 0$$

holds identically in \mathcal{M} . Therefore, since \tilde{A} is irreducible, we get

$$x_1 \frac{\partial \tilde{A}}{\partial x_1}(d, \bar{x}) + x_2 \frac{\partial \tilde{A}}{\partial x_2}(d, \bar{x}) + x_3 \frac{\partial \tilde{A}}{\partial x_3}(d, \bar{x}) = \kappa^o \tilde{A}(d, \bar{x})$$

for some constant κ^o . This implies that the polynomial $\tilde{A}(d, \bar{x})$ is homogeneous w.r.t. \bar{x} , and it follows that, for any value $d^o \notin \Delta$ (with Δ as in Corollary 1.20, page 20), $\mathcal{O}_{d^o}(\Sigma)$, has a homogeneous component. This implies that $\bar{0} \in \mathcal{O}_{d^o}(\Sigma)$ for $d^o \notin \Delta$, which is a contradiction with our hypothesis (see Remark 2.6(1), page 35). Thus, $\Theta(d, \bar{k})$ is not constant. Let us define $\Omega_{4,1}^1 = \Omega_3 \setminus \{(d^o, \bar{k}^o) / \Theta(d^o, \bar{k}^o) = 0\}$.

(3) Let us consider the following polynomials in $\mathbb{C}[d, \bar{k}, \bar{x}]$

$$\begin{cases} \sigma_0(d, \bar{k}, \bar{x}) = \det(\bar{k}, \bar{x}, \nabla_{\bar{x}} g(d, \bar{x})) \\ \sigma_1(d, \bar{k}, \bar{x}) = k_2 x_3 - k_3 x_2 \end{cases} \quad (40)$$

Let $\Omega_{4,1}^2 \subset \Omega_{4,1}^1$ be such that, for $(d^o, \bar{k}^o) \in \Omega_{4,1}^2$, these polynomials are non identically zero (note that σ_0 and σ_1 are both homogeneous w.r.t. \bar{k}). Therefore, for $(d^o, \bar{k}^o) \in \Omega_{4,1}^2$, and for $i = 0, 1$, $\sigma_i(d^o, \bar{k}^o, \bar{x})$ defines a surface $\Sigma_i(d^o, \bar{k}^o)$. From

$$\nabla_{\bar{x}} \sigma_1(\bar{k}, \bar{x}) = (0, -k_3, k_2)$$

one has

$$\nabla_{\bar{x}} g \wedge \nabla_{\bar{x}} \sigma_1 = \left(k_2 \frac{\partial g}{\partial x_2} + k_3 \frac{\partial g}{\partial x_3}, -k_2 \frac{\partial g}{\partial x_1}, -k_3 \frac{\partial g}{\partial x_1} \right).$$

Let $(d^o, \bar{k}^o) \in \Omega_{4,1}^2$ and $\bar{x}^o \in \mathcal{O}_{d^o}(\Sigma) \cap \mathcal{L}_{\bar{k}^o}$. We will show that

$$\nabla_{\bar{x}} g(d^o, \bar{x}^o) \wedge \nabla_{\bar{x}} \sigma_1(\bar{k}^o, \bar{x}^o) \neq \bar{0}. \quad (41)$$

First, note that there is $\rho^o \in \mathbb{C}$ such that $\bar{x}^o = \rho^o \bar{k}^o$. If $\frac{\partial g}{\partial x_1}(d^o, \bar{x}^o) \neq 0$, then since $k_i \neq 0$ for $i = 1, 2, 3$, the result follows. Thus, let $\frac{\partial g}{\partial x_1}(d^o, \bar{x}^o) = 0$. If we suppose that $\nabla_{\bar{x}} g(d^o, \bar{x}^o) \wedge \nabla_{\bar{x}} \sigma_1(\bar{k}^o, \bar{x}^o) = \bar{0}$, then

$$k_2^o \frac{\partial g}{\partial x_2}(d^o, \bar{x}^o) + k_3^o \frac{\partial g}{\partial x_3}(d^o, \bar{x}^o) = 0.$$

Thus, one obtains

$$\begin{cases} g(d^o, \rho^o \bar{k}^o) = 0 \\ k_1^o \frac{\partial g}{\partial x_1}(d^o, \rho^o \bar{k}^o) + k_2^o \frac{\partial g}{\partial x_2}(d^o, \rho^o \bar{k}^o) + k_3^o \frac{\partial g}{\partial x_3}(d^o, \rho^o \bar{k}^o) = 0 \end{cases}$$

and it follows that $\Theta(d^o, \bar{k}^o) = 0$ (with Θ as in step (2) of the proof), contradicting the construction of $\Omega_{4,1}^1$. Thus, Equation 41 is proved.

We will prove the analogous result for σ_0 . That is, for $(d^o, \bar{k}^o) \in \Omega_{4,1}^2$ and $\bar{x}^o \in \mathcal{O}_{d^o}(\Sigma) \cap \mathcal{L}_{\bar{k}^o}$, we will show that:

$$\nabla_{\bar{x}} g(d^o, \bar{x}^o) \wedge \nabla_{\bar{x}} \sigma_0(d^o, \bar{k}^o, \bar{x}^o) \neq \bar{0}. \quad (42)$$

From

$$\sigma_0(d, \bar{k}, \bar{x}) = \det(\bar{k}, \bar{x}, \nabla_{\bar{x}} g(d, \bar{x}))$$

and applying the derivation properties of determinants, one has, e.g.

$$\frac{\partial \sigma_0}{\partial x_1}(d, \bar{k}, \bar{x}) = \begin{vmatrix} k_1 & k_2 & k_3 \\ 1 & 0 & 0 \\ \partial_1 g & \partial_2 g & \partial_3 g \end{vmatrix} + \begin{vmatrix} k_1 & k_2 & k_3 \\ x_1 & x_2 & x_3 \\ \partial_{1,1} g & \partial_{2,1} g & \partial_{3,1} g \end{vmatrix}$$

where $\partial_i g = \frac{\partial g}{\partial x_i}$ and $\partial_{i,j} g = \frac{\partial^2 g}{\partial x_i \partial x_j}$ for $i, j \in \{1, 2, 3\}$. Let as before, $\bar{x}^o = \rho^o \bar{k}^o$ for some $\rho^o \in \mathbb{C}$. Then:

$$\frac{\partial \sigma_0}{\partial x_1}(d^o, \bar{k}^o, \bar{x}^o) = \begin{vmatrix} k_1^o & k_2^o & k_3^o \\ 1 & 0 & 0 \\ \partial_1 g & \partial_2 g & \partial_3 g \end{vmatrix} + \begin{vmatrix} k_1^o & k_2^o & k_3^o \\ \rho^o k_1^o & \rho^o k_2^o & \rho^o k_3^o \\ \partial_{1,1} g & \partial_{2,1} g & \partial_{3,1} g \end{vmatrix}$$

with all the partial derivatives evaluated at (d^o, \bar{x}^o) . Since the second determinant in the above equation vanishes, one concludes that

$$\frac{\partial \sigma_0}{\partial x_1}(d^o, \bar{k}^o, \bar{x}^o) = k_3^o \frac{\partial g}{\partial x_2}(d^o, \bar{x}^o) - k_2^o \frac{\partial g}{\partial x_3}(d^o, \bar{x}^o).$$

Similar results are obtained for the other two partial derivatives, leading to:

$$\nabla_{\bar{x}} \sigma_0(d^o, \bar{k}^o, \bar{x}^o) = \nabla_{\bar{x}} g(d^o, \bar{x}^o) \wedge \bar{k}^o.$$

Therefore,

$$\nabla_{\bar{x}} g(d^o, \bar{x}^o) \wedge \nabla_{\bar{x}} \sigma_0(d^o, \bar{k}^o, \bar{x}^o) = \nabla_{\bar{x}} g(d^o, \bar{x}^o) \wedge (\nabla_{\bar{x}} g(d^o, \bar{x}^o) \wedge \bar{k}^o).$$

Note that $\nabla_{\bar{x}} g(d^o, \bar{x}^o) \wedge \bar{k}^o \neq \bar{0}$ because, by construction, $\mathcal{L}_{\bar{k}^o}$ is not normal to $\mathcal{O}_{d^o}(\Sigma)$ at \bar{x}^o . If we suppose that

$$\nabla_{\bar{x}} g(d^o, \bar{x}^o) \wedge \nabla_{\bar{x}} \sigma_0(d^o, \bar{k}^o, \bar{x}^o) = \bar{0},$$

then the vectors $\nabla_{\bar{x}}g(d^o, \bar{x}^o)$ and $\nabla_{\bar{x}}g(d^o, \bar{x}^o) \wedge \bar{k}^o$ are parallel and perpendicular to each other. However, if two vectors are parallel and perpendicular, and one of them is not zero, then the other one must be isotropic. One concludes that $\|\nabla_{\bar{x}}g(d^o, \bar{x}^o)\| = 0$. This is a contradiction (see step (1) of the proof); therefore, Equation 42 is proved.

From Equations 41 and 42, and using Theorem 9 in [12] (page 480), we conclude that \bar{x}^o is a regular point in $\Sigma_i(d^o, \bar{k}^o) \cap \mathcal{O}_{d^o}(\Sigma)$ (for $i = 0, 1$). Besides, \bar{x}^o belongs to a unique one-dimensional component of $\Sigma_i(d^o, \bar{k}^o) \cap \mathcal{O}_{d^o}(\Sigma)$. For $i = 0, 1$, let $\mathcal{C}_i(d^o, \bar{k}^o)$ be the one-dimensional component of $\Sigma_i(d^o, \bar{k}^o) \cap \mathcal{O}_{d^o}(\Sigma)$ containing \bar{x}^o .

(4) The non-zero vector

$$\bar{v}_i(d^o, \bar{k}^o, \bar{x}^o) = \nabla_{\bar{x}}g(d^o, \bar{x}^o) \wedge \nabla_{\bar{x}}\sigma_i(\bar{k}^o, \bar{x}^o), \quad (i = 0, 1)$$

obtained in step (3) of the proof, is a tangent vector to $\mathcal{C}_i(d^o, \bar{k}^o)$ at \bar{x}^o . We will show that

$$\bar{v}_0(d^o, \bar{k}^o, \bar{x}^o) \wedge \bar{v}_1(d^o, \bar{k}^o, \bar{x}^o) \neq \bar{0}. \quad (43)$$

It holds that

$$\begin{aligned} \bar{v}_0(d^o, \bar{k}^o, \bar{x}^o) \wedge \bar{v}_1(d^o, \bar{k}^o, \bar{x}^o) = \\ -(k_3^o \frac{\partial g}{\partial x_2} - k_2^o \frac{\partial g}{\partial x_3}) \cdot \left(k_1^o \frac{\partial g}{\partial x_1} + k_2^o \frac{\partial g}{\partial x_2} + k_3^o \frac{\partial g}{\partial x_3} \right) \cdot \nabla_{\bar{x}}g(d^o, \bar{x}^o), \end{aligned}$$

with all the partial derivatives evaluated at (d^o, \bar{x}^o) . Since

$$\|\nabla f(\bar{y}^o)\| \cdot \|\nabla_{\bar{x}}g(d^o, \bar{x}^o)\| \neq 0,$$

by the fundamental property of the offset (Proposition 1.3, page 12), there is some $\kappa^o \in \mathbb{C}^\times$ such that

$$\nabla_{\bar{x}}g(d^o, \bar{x}^o) = \kappa^o \nabla f(\bar{y}^o).$$

Then, using Equation 35 (page 62) one has (see Remark 2.12, page 43) that

$$k_3^o \frac{\partial g}{\partial x_2} - k_2^o \frac{\partial g}{\partial x_3} = \kappa^o (k_3^o f_2(\bar{y}^o) - k_2^o f_3(\bar{y}^o)) = \kappa^o \frac{\beta(\bar{t}^o)}{P_0^\mu(\bar{t}^o)} (k_3^o n_2(\bar{t}^o) - k_2^o n_3(\bar{t}^o)) \neq 0.$$

Besides, in step (2) of the proof we have already seen that

$$\left(k_1^o \frac{\partial g}{\partial x_1} + k_2^o \frac{\partial g}{\partial x_2} + k_3^o \frac{\partial g}{\partial x_3} \right) \neq 0.$$

Thus, the proof of Equation 43 is finished.

(5) From Equation 38 (page 63) one has

$$M_1 \bar{k}^o, \bar{t}^o) = \epsilon \frac{d^o P_0(\bar{t}^o)}{\sqrt{h(\bar{t}^o)}} G_1(\bar{k}^o, \bar{t}^o).$$

Therefore:

$$\sqrt{h(\bar{t}^o)}(k_2^o P_3(\bar{t}^o) - k_3^o P_2(\bar{t}^o)) = d^o P_0(\bar{t}^o)(k_2(\bar{t}^o)n_3(\bar{t}^o) - k_3(\bar{t}^o)n_2(\bar{t}^o)). \quad (44)$$

Multiplying by $\frac{\beta(\bar{t}^o)}{P_0^{\mu+1}(\bar{t}^o)}$, it holds that

$$\frac{\beta(\bar{t}^o)\sqrt{h(\bar{t}^o)}}{P_0^\mu(\bar{t}^o)} \left(k_2^o \frac{P_3(\bar{t}^o)}{P_0(\bar{t}^o)} - k_3^o \frac{P_2(\bar{t}^o)}{P_0(\bar{t}^o)} \right) = d^o \left(k_2(\bar{t}^o) \frac{\beta(\bar{t}^o)n_3(\bar{t}^o)}{P_0^\mu(\bar{t}^o)} - k_3(\bar{t}^o) \frac{\beta(\bar{t}^o)n_2(\bar{t}^o)}{P_0^\mu(\bar{t}^o)} \right).$$

Using Equation 35 (page 62), one obtains (recall that $\bar{y}^o = P(\bar{t}^o)$):

$$\sqrt{h_{\text{imp}}(\bar{y}^o)}(k_2^o y_3^o - k_3^o y_2^o) - d^o(k_2^o f_3^o(\bar{y}^o) - k_3^o f_2(\bar{y}^o)) = 0. \quad (45)$$

Note also that, since $\sqrt{h(\bar{t}^o)}(k_2^o P_3(\bar{t}^o) - k_3^o P_2(\bar{t}^o)) \neq 0$ we also have

$$\sqrt{h_{\text{imp}}(\bar{y}^o)}(k_2^o y_3^o - k_3^o y_2^o) + d^o(k_2^o f_3^o(\bar{y}^o) - k_3^o f_2(\bar{y}^o)) \neq 0. \quad (46)$$

Observe that, if the sign $\epsilon = -1$ is used in the offsetting construction (see step (1) of the proof), the results in Equations 45 and 46 are reversed.

Recall (see Equation 12, page 38) that the auxiliary polynomial s_1 is given by:

$$s_1(d, \bar{k}, \bar{t}) = h(\bar{t})(k_2 P_3 - k_3 P_2)^2 - d^2 P_0(\bar{t})^2 (k_2 n_3 - k_3 n_2)^2.$$

Thus, one has:

$$\begin{aligned} & \frac{\beta^2(\bar{t})}{P_0^{2\mu+2}(\bar{t})} s_1(d, \bar{k}, \bar{t}) = \\ & \frac{\beta^2(\bar{t})}{P_0^{2\mu}(\bar{t})} \left(k_2 \frac{P_3(\bar{t})}{P_0(\bar{t})} - k_3 \frac{P_2(\bar{t})}{P_0(\bar{t})} \right)^2 - d^2 \left(k_2 \frac{\beta(\bar{t})n_3(\bar{t})}{P_0^\mu(\bar{t})} - k_3 \frac{\beta(\bar{t})n_2(\bar{t})}{P_0^\mu(\bar{t})} \right)^2. \end{aligned}$$

And substituting $\bar{y} = P(\bar{t})$ in $\frac{\beta^2(\bar{t})}{P_0^{2\mu+2}(\bar{t})} s_1(d, \bar{k}, \bar{t})$, one obtains:

$$\frac{\beta^2(\bar{t})}{P_0^{2\mu+2}(\bar{t})} s_1(d, \bar{k}, \bar{t}) = h_{\text{imp}}(\bar{y})(k_2 y_3 - k_3 y_2)^2 - d^2(k_2 f_3(\bar{y}) - k_3 f_2(\bar{y}))^2. \quad (47)$$

Let us consider the polynomial $\sigma'_1 \in \mathbb{C}[d, \bar{k}, \bar{y}]$ defined by

$$\sigma'_1(d, \bar{k}, \bar{y}) = h_{\text{imp}}(\bar{y})(k_2 y_3 - k_3 y_2)^2 - d^2(k_2 f_3(\bar{y}) - k_3 f_2(\bar{y}))^2,$$

and let $\Sigma'_1(d^o, \bar{k}^o) \subset \mathbb{C}^3$ be the algebraic closed set defined by the equation $\sigma'_1(d^o, \bar{k}^o, \bar{y}) = 0$. Let $\bar{\tau} = (\tau^1, \tau^2)$, and let $\mathcal{Q}_1(\bar{\tau})$ be a place of $\mathcal{T}_1^a(d^o, \bar{k}^o)$ centered at \bar{t}^o . We assume that $\mathcal{Q}_1(\bar{0}) = \bar{t}^o$. Since $T_1(d^o, \bar{k}^o, 1, \mathcal{Q}_1(\bar{\tau})) = 0$ identically in $\bar{\tau}$, from Equation 28 (page 51) it follows that

$$s_1(d^o, \bar{k}^o, \mathcal{Q}_1(\bar{\tau})) = S_1(d^o, \bar{k}^o, 1, \mathcal{Q}_1(\bar{\tau})) = 0$$

identically in $\bar{\tau}$. Thus, from Equation 47 (recall that $\bar{y} = P(\bar{t})$ in the lhs of Equation 47) one has:

$$\sigma'_1(d^o, \bar{k}^o, P(\mathcal{Q}_1(\bar{\tau}))) = 0$$

identically in $\bar{\tau}$. Note that:

$$\sigma'_1(d, \bar{k}^o, \bar{y}) = \sigma'_{1,+}(d, \bar{k}^o, \bar{y})\sigma'_{1,-}(d, \bar{k}^o, \bar{y})$$

with

$$\begin{cases} \sigma'_{1,+}(d, \bar{k}^o, \bar{y}) = \sqrt{h_{\text{imp}}(\bar{y})}(k_2^o y_3 - k_3^o y_2) + d(k_2^o f_3(\bar{y}) - k_3^o f_2(\bar{y})), \\ \sigma'_{1,-}(d, \bar{k}^o, \bar{y}) = \sqrt{h_{\text{imp}}(\bar{y})}(k_2^o y_3 - k_3^o y_2) - d(k_2^o f_3(\bar{y}) - k_3^o f_2(\bar{y})). \end{cases}$$

The functions $\sigma'_{1,+}(d, \bar{k}^o, \bar{y})$ and $\sigma'_{1,-}(d, \bar{k}^o, \bar{y})$ are analytic in the neighborhood U^1 of (d^o, \bar{x}^o) introduced in step (1) of the proof. Therefore:

$$\sigma'_{1,+}(d^o, \bar{k}^o, \mathcal{Q}_1(\bar{\tau}))\sigma'_{1,-}(d^o, \bar{k}^o, \mathcal{Q}_1(\bar{\tau})) = 0,$$

identically in $\bar{\tau}$. However, evaluating at $\bar{\tau} = \bar{0}$, and taking Equations 45 and 46 into account, one sees that

$$\sigma'_{1,+}(d, \bar{k}^o, \bar{y}^o) \neq 0, \text{ while } \sigma'_{1,-}(d, \bar{k}^o, \bar{y}^o) = 0.$$

By the analytic character of the functions, one concludes that

$$\sigma'_{1,-}(d^o, \bar{k}^o, \mathcal{Q}_1(\bar{\tau})) = 0,$$

identically in $\bar{\tau}$. Dividing by $\sqrt{h_{\text{imp}}(\bar{y})}$, this relation implies that:

$$k_2^o \left(\frac{P_3(\mathcal{Q}_1(\bar{\tau}))}{P_0(\mathcal{Q}_1(\bar{\tau}))} + d^o \frac{f_3(P(\mathcal{Q}_1(\bar{\tau})))}{\sqrt{h_{\text{imp}}(P_3(\mathcal{Q}_1(\bar{\tau})))}} \right) - k_3^o \left(\frac{P_2(\mathcal{Q}_1(\bar{\tau}))}{P_0(\mathcal{Q}_1(\bar{\tau}))} + d^o \frac{f_2(P(\mathcal{Q}_1(\bar{\tau})))}{\sqrt{h_{\text{imp}}(P_2(\mathcal{Q}_1(\bar{\tau})))}} \right) = 0.$$

That is,

$$k_3^o \varphi_2(d^o, P(\mathcal{Q}_1(\bar{\tau}))) - k_2^o \varphi_3(d^o, P(\mathcal{Q}_1(\bar{\tau}))) = 0$$

identically in $\bar{\tau}$, where $\bar{\varphi} = (\varphi_2, \varphi_3)$ was defined in step (1) of the proof (see Equation 37, page 63). With the notation introduced in step (3) of the proof (see Equation 40, page 65), this is

$$\sigma_1(d^o, \bar{\varphi}(d^o, P(\mathcal{Q}_1(\bar{\tau})))) = 0,$$

identically in $\bar{\tau}$. This implies that if \mathcal{B}_1 is the branch of $\mathcal{T}_1^a(d^o, \bar{k}^o)$ at \bar{t}^o determined by $\mathcal{Q}_1(\bar{\tau})$, then

$$\bar{\nu}_{d^o}(\mathcal{B}_1) \subset \mathcal{C}_1(d^o, \bar{k}^o),$$

where $\bar{\nu}_{d^o}$ was defined in Equation 39 (page 64), and $\mathcal{C}_1(d^o, \bar{k}^o)$ was introduced at the end of step (3) of the proof.

- (6) Let $\mathcal{Q}_0(\bar{\tau})$ be a place of $\mathcal{T}_0^a(\bar{k}^o)$ centered at \bar{t}^o . We assume that $\mathcal{Q}_0(\bar{0}) = \bar{t}^o$. Since $T_0(\bar{k}^o, 1, \mathcal{Q}_0(\bar{\tau})) = 0$ identically in $\bar{\tau}$, from Equation 27 (page 51) it follows that

$$s_0(\bar{k}^o, \mathcal{Q}_0(\bar{\tau})) = S_0(\bar{k}^o, 1, \mathcal{Q}_0(\bar{\tau})) = 0,$$

identically in $\bar{\tau}$. That is,

$$s_0(\bar{k}^o, \mathcal{Q}_0(\bar{\tau})) = \det \begin{pmatrix} k_1^o & k_2^o & k_3^o \\ P_1(\mathcal{Q}_0(\bar{\tau})) & P_2(\mathcal{Q}_0(\bar{\tau})) & P_3(\mathcal{Q}_0(\bar{\tau})) \\ n_1(\mathcal{Q}_0(\bar{\tau})) & n_2(\mathcal{Q}_0(\bar{\tau})) & n_3(\mathcal{Q}_0(\bar{\tau})) \end{pmatrix}, \quad (48)$$

identically in $\bar{\tau}$. Multiplying this by $\frac{\beta(\mathcal{Q}_0(\bar{\tau}))}{P_0^{\mu+1}(\mathcal{Q}_0(\bar{\tau}))}$, one has:

$$\det \begin{pmatrix} k_1^o & k_2^o & k_3^o \\ \frac{P_1(\mathcal{Q}_0(\bar{\tau}))}{P_0(\bar{\tau})} & \frac{P_2(\mathcal{Q}_0(\bar{\tau}))}{P_0(\mathcal{Q}_0(\bar{\tau}))} & \frac{P_3(\mathcal{Q}_0(\bar{\tau}))}{P_0(\mathcal{Q}_0(\bar{\tau}))} \\ \frac{\beta(\mathcal{Q}_0(\bar{\tau}))n_1(\mathcal{Q}_0(\bar{\tau}))}{P_0^\mu(\mathcal{Q}_0(\bar{\tau}))} & \frac{\beta(\bar{\tau})n_2(\mathcal{Q}_0(\bar{\tau}))}{P_0^\mu(\mathcal{Q}_0(\bar{\tau}))} & \frac{\beta(\bar{\tau})n_3(\mathcal{Q}_0(\bar{\tau}))}{P_0^\mu(\mathcal{Q}_0(\bar{\tau}))} \end{pmatrix} = 0,$$

identically in $\bar{\tau}$. Using Equation 35 (page 62), this implies that:

$$\det(\bar{k}^o, P(\mathcal{Q}_0(\bar{\tau})), \nabla f(P(\mathcal{Q}_0(\bar{\tau})))) = 0.$$

Since

$$\bar{\varphi}(d^o, P(\mathcal{Q}_0(\bar{\tau}))) = P(\mathcal{Q}_0(\bar{\tau})) \pm d^o \frac{\nabla f(P(\mathcal{Q}_0(\bar{\tau})))}{\sqrt{h_{\text{imp}}(P(\mathcal{Q}_0(\bar{\tau})))}}$$

and the second term in the sum is parallel to $\nabla f(P(\mathcal{Q}_0(\bar{\tau})))$, we have

$$\det(\bar{k}^o, \bar{\varphi}(d^o, P(\mathcal{Q}_0(\bar{\tau}))), \nabla f(P(\mathcal{Q}_0(\bar{\tau})))) = 0.$$

Besides, by the fundamental property of the offset (Proposition 1.3, page 12), and the construction in step (1) of the proof, the vectors

$$\nabla f(y) \text{ and } \nabla_{\bar{x}} g(d^o, \bar{\varphi}(d^o, \bar{y}))$$

are parallel for every value of (d^o, \bar{y}) in V^1 . It follows that

$$\det(\bar{k}^o, \bar{\varphi}(d^o, P(\mathcal{Q}_0(\bar{\tau}))), \nabla_{\bar{x}} g(d^o, \bar{\varphi}(d^o, P(\mathcal{Q}_0(\bar{\tau})))) = 0,$$

identically in $\bar{\tau}$. Recalling the definition of σ_0 in Equation 40 (page 65), this implies that

$$\sigma_0(d^o, \bar{k}^o, \bar{\varphi}(d^o, P(\mathcal{Q}_0(\bar{\tau})))) = 0,$$

identically in $\bar{\tau}$. It follows that, if \mathcal{B}_0 is the branch of $\mathcal{T}_0^a(\bar{k}^o)$ at \bar{t}^o determined by $\mathcal{Q}_0(\bar{\tau})$, then

$$\bar{\nu}_{d^o}(\mathcal{B}_0) \subset \mathcal{C}_0(d^o, \bar{k}^o),$$

where $\bar{\nu}_{d^o}$ was defined in Equation 39 (page 64), and $\mathcal{C}_0(d^o, \bar{k}^o)$ was introduced at the end of step (3) of the proof.

Now we can finish the proof of the proposition. In steps (5) and (6) of the proof we have shown that any branch at \bar{t}^o of the curves $\mathcal{T}_0^a(\bar{k}^o)$ or $\mathcal{T}_1^a(d^o, \bar{k}^o)$ is mapped by $\bar{\nu}_{d^o}$ respectively into the curve $\mathcal{C}_1(d^o, \bar{k}^o)$ or $\mathcal{C}_0(d^o, \bar{k}^o)$ (these curves are constructed in step (3)). Since $d\bar{\nu}_{d^o}$ is an isomorphism of vector spaces (see step (1)), it follows that:

- By the results in step (3), there is only one branch at \bar{t}^o of each of the curves $\mathcal{T}_0^a(\bar{k}^o)$ and $\mathcal{T}_1^a(d^o, \bar{k}^o)$. Besides, since the rank of the Jacobian matrix (and therefore, the condition in [12], Theorem 9, page 480) is preserved under analytic isomorphisms, the unique branch of each the curves $\mathcal{T}_0^a(\bar{k}^o)$ and $\mathcal{T}_1^a(d^o, \bar{k}^o)$ passing through \bar{t}^o is regular at that point.
- By the results in step (4), if ℓ_1 and ℓ_0 are the two tangent lines of these two branches, then ℓ_1 and ℓ_0 are different.

Then

$$\text{mult}_{\bar{t}^o}(\mathcal{T}_0, \mathcal{T}_1) = 1$$

follows from Theorem 5.10 in [43] (page 114). \square

3.4 The degree formula

Before the statement of the degree formula we need to introduce some more notation and a technical lemma. Let

$$R(\bar{c}, d, \bar{k}, \bar{t}) = \text{Res}_{t_0}(T_0(\bar{k}, \bar{t}_h), T(\bar{c}, d, \bar{k}, \bar{t}_h))$$

(for the definition of T_0 and T see Equations 27 and 29, in page 51). Then R factors as follows:

$$R(\bar{c}, d, \bar{k}, \bar{t}) = N_1(d, \bar{k}, \bar{t})M_3(\bar{c}, d, \bar{k}, \bar{t})$$

where $N_1(d, \bar{k}, \bar{t}) = \text{Con}_{\bar{c}}(R)$ and $M_3(\bar{c}, d, \bar{k}, \bar{t}) = \text{PP}_{\bar{c}}(R)$.

Besides, N_1 factors as follows:

$$N_1(d, \bar{k}, \bar{t}) = M_1(\bar{t})M_2(d, \bar{k}, \bar{t})$$

where $M_1(\bar{t}) = \text{Con}_{(d, \bar{k})}(N_1)$ and $M_2(d, \bar{k}, \bar{t}) = \text{PP}_{(d, \bar{k})}(N_1)$. Thus, one has

$$R(\bar{c}, d, \bar{k}, \bar{t}) = M_1(\bar{t})M_2(d, \bar{k}, \bar{t})M_3(\bar{c}, d, \bar{k}, \bar{t})$$

and

$$M_2(d, \bar{k}, \bar{t}) = \text{PP}_{(d, \bar{k})}(\text{Con}_{\bar{c}}(R)).$$

Note that M_1, M_2 and M_3 are homogeneous polynomials in $\bar{t} = (t_1, t_2)$.

The following lemma deals with the specialization of the resultant $R(\bar{c}, d, \bar{k}, \bar{t})$. More precisely, for $(d^o, \bar{k}^o) \in \mathbb{C} \times \mathbb{C}^3$ we denote:

$$T_0^{\bar{k}^o}(\bar{t}_h) = T_0(d^o, \bar{k}^o, \bar{t}_h), \quad T^{(d^o, \bar{k}^o)}(\bar{c}, \bar{t}_h) = T(\bar{c}, d^o, \bar{k}^o, \bar{t}_h).$$

and

$$R^{(d^o, \bar{k}^o)}(\bar{c}, \bar{t}) = \text{Res}_{t_0} \left(T_0^{\bar{k}^o}(\bar{t}_h), T^{(d^o, \bar{k}^o)}(\bar{c}, \bar{t}_h) \right)$$

Lemma 3.21. *Let Ω_4 be as in Proposition 3.20 (page 62). There exists a non-empty open Ω_5 , such that for $(d^o, \bar{k}^o) \in \Omega_5$ the following hold:*

(a) *The resultant $R(\bar{c}, d, \bar{k}, \bar{t})$ specializes properly:*

$$R^{(d^o, \bar{k}^o)}(\bar{c}, \bar{t}) = R(\bar{c}, d^o, \bar{k}^o, \bar{t}) = M_1(\bar{t})M_2(d^o, \bar{k}^o, \bar{t})M_3(\bar{c}, d^o, \bar{k}^o, \bar{t}).$$

(b) *The content w.r.t \bar{c} also specializes properly:*

$$\text{Con}_{\bar{c}}(R^{(d^o, \bar{k}^o)})(\bar{t}) = M_1(\bar{t})M_2(d^o, \bar{k}^o, \bar{t}).$$

(c) *the coprimality of M_1 and M_2 is invariant under specialization:*

$$\gcd(M_1(\bar{t}), M_2(d^o, \bar{k}^o, \bar{t})) = 1$$

Proof. For (a), consider T_0 and T as polynomials in $\mathbb{C}[\bar{c}, d, \bar{k}, \bar{t}][t_0]$. Let $\text{lc}(T_0)(\bar{k}, \bar{t})$, (resp. $\text{lc}(T)(\bar{c}, d, \bar{k}, \bar{t})$) be a leading coefficient w.r.t. t_0 of T_0 (resp. T). Take $A_1(\bar{k})$ (resp. $B_1(d, \bar{k})$) to be the coefficient of a term of $\text{lc}(T_0)(\bar{k}, \bar{t})$ (resp. $\text{lc}(T)(d, \bar{k}, \bar{t})$) of degree equal to $\deg_{\bar{t}}(\text{lc}(T_0)(\bar{k}, \bar{t}))$ (resp. $\deg_{\{\bar{c}, \bar{t}\}}(\text{lc}(T)(\bar{k}, \bar{t}))$). Now, if $A_1(\bar{k}^o)B_1(d^o, \bar{k}^o) \neq 0$, then (a) holds. Thus, set

$$\Omega_5^1 = \Omega_4 \cap \{(d^o, \bar{k}^o)/A_1(\bar{k}^o)B_1(d^o, \bar{k}^o) \neq 0\}.$$

For (b), we know that $M_3(\bar{c}, d, \bar{k}, \bar{t})$ is primitive w.r.t. \bar{c} . If $M_3(\bar{c}, d, \bar{k}, \bar{t})$ (considered as a polynomial in $\mathbb{C}[d, \bar{k}, \bar{t}][\bar{c}]$) has only one term, then its coefficient w.r.t. \bar{c} must be

constant, and so M_3 remains primitive under specialization of (d, \bar{k}) . Suppose, on the other hand, that $M_3(\bar{c}, d, \bar{k}, \bar{t})$ has more than one term, and let:

$$M_{3,1}(d, \bar{k}, \bar{t}), \dots, M_{3,\rho}(d, \bar{k}, \bar{t})$$

be an (arbitrary) ordering of its non-zero coefficients w.r.t. \bar{c} . Let $\Gamma_1(d, \bar{k}, \bar{t}) = M_{3,1}(d, \bar{k}, \bar{t})$, and for $j = 2, \dots, \rho$ let

$$\Gamma_j(d, \bar{k}, \bar{t}) = \gcd(M_{3,j}(d, \bar{k}, \bar{t}), \Gamma_{j-1}(d, \bar{k}, \bar{t}))$$

Note that, for $j = 1, \dots, \rho$, the $M_{3,j}$ are homogeneous in \bar{t} of the same degree. Thus, the Γ_j are either homogeneous in \bar{t} , or they only depend on (d, \bar{k}) .

Since M_3 is primitive w.r.t. \bar{c} , let j° be the first index value in $1, \dots, \rho$ for which $\Gamma_{j^\circ}(d, \bar{k}, \bar{t}) = 1$. If $j^\circ = 1$, then $M_{3,1}(d, \bar{k}, \bar{t})$ is a constant, and in this case it is obvious that M_3 remains primitive under specialization of (d, \bar{k}) . If $j^\circ > 1$, we consider:

$$\text{Res}_{t_1}(M_{3,j^\circ}(d, \bar{k}, \bar{t}), \Gamma_{j^\circ-1}(d, \bar{k}, \bar{t}))$$

This resultant is not identically zero, because we have assumed that $\Gamma_{j^\circ-1}(d, \bar{k}, \bar{t}) = 1$. Since the involved polynomials are homogeneous in \bar{t} , this resultant is of the form $t_2^p \Phi(d, \bar{k})$ for some $p \in \mathbb{N}$ and some $\Phi \in \mathbb{C}[d, \bar{k}]$. Now, because of the construction, if $\Phi(d^\circ, \bar{k}^\circ) \neq 0$, the specialization $M_3(\bar{c}, d^\circ, \bar{k}^\circ, \bar{t})$ is primitive w.r.t. \bar{c} . Thus, set:

$$\Omega_5^2 = \Omega_5^1 \cap \{(d^\circ, \bar{k}^\circ) / \Phi(d^\circ, \bar{k}^\circ) \neq 0\}.$$

For (c) we use a similar construction. If either M_1 or M_2 do not depend on \bar{t}_h , the result is trivial. Otherwise, M_1 and M_2 are both homogeneous polynomials in \bar{t} , so the resultant

$$\text{Res}_{t_1}(M_1(\bar{t}), M_2(d, \bar{k}, \bar{t}))$$

is of the form $t_2^{\tilde{p}} \tilde{\Phi}(d, \bar{k})$ for some $\tilde{p} \in \mathbb{N}$ and some $\tilde{\Phi}_1 \in \mathbb{C}[d, \bar{k}]$. Thus, if $\tilde{\Phi}_1(d^\circ, \bar{k}^\circ) \neq 0$, then $M_1(\bar{t})$ and $M_2(d^\circ, \bar{k}^\circ, \bar{t})$ do not have common factors of positive degree in t_1 . A similar construction can be carried out w.r.t. t_2 , obtaining a certain $\tilde{\Phi}_2$. Thus, set:

$$\Omega_5^3 = \Omega_5^2 \cap \{(d^\circ, \bar{k}^\circ) / \tilde{\Phi}_1(d^\circ, \bar{k}^\circ) \tilde{\Phi}_2(d^\circ, \bar{k}^\circ) \neq 0\}.$$

The construction shows that the lemma holds for $\Omega_5 = \Omega_5^3$. □

Finally, we are ready to state and prove the degree formula.

Theorem 3.22 (Total Degree Formula for the Offset of a Parametric Surface). *Let T_0 and T be as in Equations 27 and 29 (page 51). Then:*

$$m \cdot \delta = \deg_{\{\bar{t}\}} \left(\text{PP}_{(d, \bar{k})} \left(\text{Con}_{\bar{c}} \left(\text{Res}_{t_0} \left(T_0(\bar{k}, \bar{t}_h), T(\bar{c}, d, \bar{k}, \bar{t}_h) \right) \right) \right) \right) = \deg_{\bar{t}}(M_2(d, \bar{k}, \bar{t}))$$

where m is the tracing index of P (see Remark 1.25, page 24), and if $g(d, \bar{x})$ is the generic offset polynomial of Σ , then $\delta = \deg_{\bar{x}}(g(d, \bar{x}))$.

Proof. Recall that (see Remark 3.11, page 56), since $\Omega_3 \subset \Omega_2$, if $(d^o, \bar{k}^o) \in \Omega_3$ it holds that

$$m\delta = \# \left(\mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o) \right).$$

Thus, to prove the theorem it suffices to show that for any of these (d^o, \bar{k}^o) , it holds that

$$\deg_{\bar{t}}(M_2(d, \bar{k}, \bar{t})) = \# \left(\mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o) \right).$$

In order to do this, we will specialize at (d^o, \bar{k}^o) . More specifically, we will show that there is an open non-empty subset $\Omega_6 \subset \Omega_5$, such that, if $(d^o, \bar{k}^o) \in \Omega_6$, then $\deg_{\bar{t}}(M_2(d^o, \bar{k}^o, \bar{t}))$ equals $\# \left(\mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o) \right)$.

Let Ω_5 be as in Lemma 3.21 (page 72), and let $(d^o, \bar{k}^o) \in \Omega_5$. Note that $M_1(\bar{t})$ and $M_2(d^o, \bar{k}^o, \bar{t})$ both factor as product of linear factors. there exists $\gamma \in \mathbb{C}$ such that $(\gamma : \alpha : \beta) \in \Psi_5^P(d^o, \bar{k}^o)$. Let us see that if $M_1(\bar{t}^o) = 0$, with $\bar{t}^o = (t_1^o, t_2^o)$, and there is t_0^o such that $(t_0^o : t_1^o : t_2^o) \in \Psi_5^P(d^o, \bar{k}^o)$, then $\bar{t}_h^o \notin \mathcal{A}_h$. In fact, if $t_0^o = 0$, the result follows from Proposition 3.9 (page 53) and Proposition 3.12 (page 56). Thus, w.l.o.g we suppose that $\bar{t}_h^o = (1 : t_1^o : t_2^o)$, with $\bar{t}_h^o \in \mathcal{A}_h$. Then using Proposition 3.12 (page 56), we get $\bar{t}_h^o \notin \mathcal{I}_5^{P_h}(\Omega_2)$. This is a contradiction, since $M_1(\bar{t})$ does not depend on (d, \bar{k}) .

We will now show that there is an open set $\Omega_6 \subset \Omega_5$ such that if $(d^o, \bar{k}^o) \in \Omega_6$ and $M_2(d^o, \bar{k}^o, \bar{t}^o) = 0$, then there is t_0^o such that $\bar{t}_h^o = (t_0^o, \bar{t}^o) \in \Psi_5^P(d^o, \bar{k}^o)$. This follows from Lemma 1.33, page 28. Let us define:

$$\begin{cases} \Omega_6^1 = \Omega_5 \cap \{(d^o, \bar{k}^o) \mid \deg_{\bar{t}_0}(T_i(d^o, \bar{k}^o, \bar{t}_h)) = \deg_{\bar{t}_0}(T_i(d, \bar{k}, \bar{t}_h)) \text{ for } i = 1, 2, 3\} \\ \Omega_6^2 = \Omega_6^1 \cap \{(d^o, \bar{k}^o) \mid \deg_{\bar{t}_h}(T_i(d^o, \bar{k}^o, \bar{t}_h)) = \deg_{\bar{t}_h}(T_i(d, \bar{k}, \bar{t}_h)) \text{ for } i = 1, 2, 3\} \\ \Omega_6^3 = \Omega_6^2 \cap \{(d^o, \bar{k}^o) \mid \gcd(T_1(d^o, \bar{k}^o, \bar{t}_h), T_2(d^o, \bar{k}^o, \bar{t}_h), T_3(d^o, \bar{k}^o, \bar{t}_h)) = 1\} \end{cases}$$

The sets Ω_6^1 and Ω_6^2 are open and non-empty because they are defined by the non-vanishing of the corresponding leading coefficients. The fact that Ω_6^3 is open and non-empty follows from a similar argument to the proof of Lemma 3.21(c) (page 72). Finally, take $\Omega_6 = \Omega_6^3$. Then, (i), (ii) and (iii) in Lemma 1.33 hold because of the construction of Ω_6^i for $i = 1, 2, 3$, respectively. And also

$$\text{lc}_{t_0}(T_0)(\bar{t}^o) \cdot \text{lc}_{t_0}(T)(\bar{c}, \bar{t}^o) \neq 0$$

holds because of the construction of Ω_5^1 in Lemma 3.21 (page 72), and because $\Omega_6 \subset \Omega_5$.

Let $(d^o, \bar{k}^o) \in \Omega_6$. If $\bar{t}_h^o \in \mathcal{A} \cap \Psi_5^{P_h}(d^o, \bar{k}^o)$, then $M_1(\bar{t}^o)M_2(d^o, \bar{k}^o, \bar{t}^o) = 0$. Since we have seen that $M_1(\bar{t}^o) \neq 0$, one concludes that $M_2(d^o, \bar{k}^o, \bar{t}^o) = 0$. Conversely, let \bar{t}^o be such that $M_2(d^o, \bar{k}^o, \bar{t}^o) = 0$. Then, by the construction of Ω_6 , there is t_0^o such that $(t_0^o : \bar{t}^o) \in \Psi_5^P(d^o, \bar{k}^o)$. Let us see that $\bar{t}_h^o \in \mathcal{A}_h$. If $\bar{t}_h^o \in \mathcal{I}_5^{P_h}(\Omega_2)$, then because of the invariance, $M_1(\bar{t}^o) = 0$, and this contradicts Lemma 3.21(c) (page 72). Thus, $\bar{t}_h^o \notin \mathcal{I}_5^{P_h}(\Omega_2)$, and by Proposition 3.17 (page 61), one has $\bar{t}_h^o \in \mathcal{A}_h$.

Thus, we have shown that for each of the factors of $M_2(d^o, \bar{k}^o, \bar{t})$ there is a point $\bar{t}_h^o \in \mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o)$ such that $M_2(d^o, \bar{k}^o, \bar{t}^o) = 0$, and conversely. Let $L^{(\alpha, \beta)}(\bar{t}) = \beta t_1 - \alpha t_2$ be one of these factors of $M_2(d^o, \bar{k}^o, \bar{t})$, and let $\mathcal{L}^{(\alpha, \beta)}$ the line defined by the equation $L^{(\alpha, \beta)}(\bar{t}) = 0$. By Lemma 3.21(c) (page 72), one has

$$\# \left(\mathcal{L}^{(\alpha, \beta)} \cap \mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o) \right) = \# \left(\mathcal{L}^{(\alpha, \beta)} \cap \Psi_5^{P_h}(d^o, \bar{k}^o) \right). \quad (49)$$

If we define

$$p(\alpha, \beta) = \# \left(\mathcal{L}^{(\alpha, \beta)} \cap \Psi_5^{P_h}(d^o, \bar{k}^o) \right),$$

then we will show that $L^{(\alpha, \beta)}(\bar{t})$ appears in $M_2(d^o, \bar{k}^o, \bar{t})$ with exponent equal to $p(\alpha, \beta)$. From this it will follow that:

$$\# \left(\mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o) \right) = \sum_{(\alpha, \beta)} \# \left(\mathcal{L}^{(\alpha, \beta)} \cap \Psi_5^{P_h}(d^o, \bar{k}^o) \right) = \sum_{(\alpha, \beta)} p(\alpha, \beta) = \deg_{\bar{t}} (M_2(d, \bar{k}, \bar{t})),$$

and this will conclude the proof of the theorem.

To prove our claim, note that $\mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o)$ is a finite set, and by Proposition 3.20 (page 62), if $\bar{t}_h^o \in \mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o)$, then:

$$\min_{i=1,2,3} (\text{mult}_{\bar{t}_h^o}(\mathcal{T}_o(\bar{k}^o), \mathcal{T}_i(d^o, \bar{k}^o))) = 1. \quad (50)$$

Recall that

$$T_0^{\bar{k}^o}(\bar{t}_h) = T_0(d^o, \bar{k}^o, \bar{t}_h)$$

and

$$T^{(d^o, \bar{k}^o)}(\bar{c}, \bar{t}_h) = T(\bar{c}, d^o, \bar{k}^o, \bar{t}_h) = c_1 T_1(d^o, \bar{k}^o, \bar{t}_h) + c_2 T_2(d^o, \bar{k}^o, \bar{t}_h) + c_3 T_3(d^o, \bar{k}^o, \bar{t}_h).$$

For $\bar{c}^o \in \mathbb{C}^3$, let $\mathcal{T}^{(\bar{c}^o, d^o, \bar{k}^o)}$ be the algebraic closed subset of \mathbb{P}^2 defined by the equation $T^{(d^o, \bar{k}^o)}(\bar{c}^o, \bar{t}_h) = 0$. Note that there is an open set of values \bar{c}^o for which $\mathcal{T}^{(\bar{c}^o, d^o, \bar{k}^o)}$ is indeed a curve. Let us see that there is an open subset $A(\bar{t}_h^o) \subset \mathbb{C}^3$, such that if $\bar{c}^o \in A(\bar{t}_h^o)$, then

$$\text{mult}_{\bar{t}_h^o}(\mathcal{T}_0^{\bar{k}^o}, \mathcal{T}^{(\bar{c}^o, d^o, \bar{k}^o)}) = 1.$$

To prove this, let $\mathcal{P}(\bar{\tau})$ be a place of $\mathcal{T}_0^{\bar{k}^o}$ at \bar{t}_h^o . Then, by Equation 50 the order of the power series $T^{(d^o, \bar{k}^o)}(\bar{c}, \mathcal{P}(\bar{\tau}))$ is one. From this, one sees that it suffices to take $A(\bar{t}_h^o)$ to be the open set of values \bar{c}^o for which the order of $T^{(d^o, \bar{k}^o)}(\bar{c}, \mathcal{P}(\bar{\tau}))$ does not increase.

Let now

$$\bar{c}^o \in \bigcap_{\bar{t}_h^o \in \mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o)} A(\bar{t}_h^o).$$

Applying Lemma 1.32 (page 28) to the curves $\mathcal{T}_0^{\bar{k}^o}$ and $\mathcal{T}^{(\bar{c}^o, d^o, \bar{k}^o)}$, and the line $\mathcal{L}^{(\alpha, \beta)}$, one concludes that the factor $\beta t_1 - \alpha_j t_2$ appears in $\text{Res}_{t_0} \left(T_0^{\bar{k}^o}(\bar{t}_h), T^{(d^o, \bar{k}^o)}(\bar{c}^o, \bar{t}_h) \right)$ with exponent equal to:

$$\sum_{\bar{t}_h^o \in \mathcal{L}^{(\alpha, \beta)} \cap \mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o)} \text{mult}_{\bar{t}_h^o}(\mathcal{T}_0^{\bar{k}^o}, \mathcal{T}^{(\bar{c}^o, d^o, \bar{k}^o)}) = \#(\mathcal{L}^{(\alpha, \beta)} \cap \mathcal{A}_h \cap \Psi_5^{P_h}(d^o, \bar{k}^o)).$$

Taking Equation 49 into account, this finishes the proof of our claim, and of the theorem. \square

We will finish this section with some examples, illustrating the use of the degree formula in Theorem 3.22 (page 73). The implicit equations in these examples have been obtained with the Computer Algebra System CoCoA (see [11]).

Example 3.23. Let Σ be the surface (a hyperbolic paraboloid) with implicit equation

$$y_3 - y_1^2 + \frac{y_2^2}{4} = 0.$$

A rational –in fact polynomial– parametrization of Σ is given by:

$$P(t_1, t_2) = (t_1, 2t_2, t_1^2 - t_2^2).$$

From the form of its two first components, it is clear that this is a proper parametrization. This surface and its offset at $d^o = 1$ are illustrated in Figure 3. The homogeneous associated normal vector is

$$N(\bar{t}_h) = (-2t_1, t_2, t_0).$$

Then the auxiliary curves are:

$$T_0(\bar{t}_h) = 2k_1 t_2 t_0^2 - k_1 t_2 t_1^2 + k_1 t_2^3 - t_1 t_0^2 k_2 + 5t_1 t_0 k_3 t_2 - 2k_2 t_1^3 + 2t_1 k_2 t_2^2,$$

$$T_1(\bar{t}_h) = 4t_1^6 k_2^2 - 7t_1^4 k_2^2 t_2^2 - 16t_1^4 k_2 k_3 t_2 t_0 + 2t_1^2 k_2^2 t_2^4 + 12t_1^2 k_2 t_2^3 k_3 t_0 + 16t_1^2 k_3^2 t_2^2 t_0^2 + t_2^6 k_2^2 + 4t_2^5 k_2 k_3 t_0 + 4t_2^4 k_3^2 t_0^2 + t_0^2 k_2^2 t_1^4 - 2t_0^2 k_2^2 t_1^2 t_2^2 - 4t_0^3 k_2 t_1^2 k_3 t_2 + t_0^2 k_2^2 t_2^4 + 4t_0^3 k_2 t_2^3 k_3 + 4t_0^4 k_3^2 t_2^2 - d^2 t_0^6 k_2^2 + 2d^2 t_0^5 k_2 k_3 t_2 - d^2 t_0^4 k_3^2 t_2^2,$$

$$T_2(\bar{t}_h) = 4t_1^4 t_0^2 k_3^2 - 8t_1^5 t_0 k_3 k_1 + 6t_1^3 t_0 k_3 k_1 t_2^2 + 4t_1^6 k_1^2 - 7t_1^4 k_1^2 t_2^2 + 2t_1^2 k_1^2 t_2^4 + t_1^2 k_3^2 t_2^2 t_0^2 + 2t_1^4 t_0 k_3 k_1 + t_2^6 k_1^2 + t_0^4 t_1^2 k_3^2 - 2t_0^3 t_1^3 k_3 k_1 + 2t_0^3 t_1 k_3 k_1 t_2^2 + t_0^2 k_1^2 t_1^4 - 2t_0^2 k_1^2 t_1^2 t_2^2 + t_0^2 k_1^2 t_2^4 - 4d^2 t_0^4 t_1^2 k_3^2 - 4d^2 t_0^5 k_3 k_1 t_1 - d^2 t_0^6 k_1^2,$$

$$T_3(\bar{t}_h) = t_0^2(4k_2^2 t_1^4 - 16t_1^3 k_2 k_1 t_2 + 16k_1^2 t_1^2 t_2^2 + k_2^2 t_1^2 t_2^2 - 4t_2^3 k_2 t_1 k_1 + 4k_1^2 t_2^4 + t_0^2 k_2^2 t_1^2 - 4k_2 t_1 t_0^2 k_1 t_2 + 4k_1^2 t_2^2 t_0^2 - 4t_0^2 d^2 k_2^2 t_1^2 - 4t_0^2 d^2 t_1 k_2 k_1 t_2 - t_0^2 d^2 k_1^2 t_2^2).$$

Denoting, as in the degree formula,

$$R(\bar{c}, d, \bar{k}, \bar{t}) = \text{Res}_{t_0} (T_0(\bar{k}, \bar{t}_h), T(\bar{c}, d, \bar{k}, \bar{t}_h)),$$

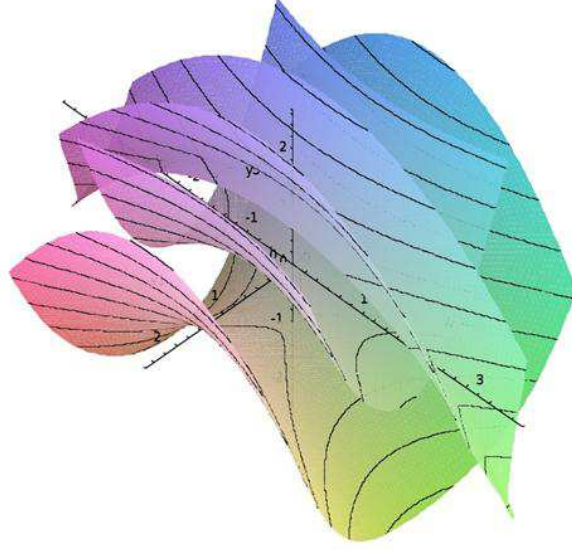


Figure 3: Hyperbolic paraboloid and one of its offsets

one has:

$$\begin{aligned}
 R(\bar{c}, d, \bar{k}, \bar{t}) = & (t_1 - t_2)^2 (t_1 + t_2)^2 (4 t_2^4 c_3^2 k_1^4 + t_2^2 c_2^2 k_1^4 + t_2^2 c_1^2 k_2^4 + t_1^2 t_2^2 c_3^2 k_1^2 k_2^2 + 4 t_1^2 t_2^2 c_1 c_3 k_1^2 k_2^2 + \\
 & 4 t_1^2 t_2^2 c_2 c_3 k_1^2 k_2^2 + 4 t_1^2 t_2^2 c_2 c_3 k_1^4 + 4 t_1^2 t_2^2 c_1 c_3 k_2^4 + 4 t_1^2 t_2^2 c_1 c_2 k_2^2 k_3^2 + 4 t_1^2 t_2^2 c_1 c_2 k_1^2 k_3^2 + \\
 & 17 t_1^2 t_2^2 c_1 c_3 k_2^2 k_3^2 + 17 t_1^2 t_2^2 c_2 c_3 k_1^2 k_3^2 - 4 t_1^2 t_2^2 c_1^2 k_2^2 k_3^2 + 17 t_1^2 c_2 c_1 k_3^4 t_2^2 - 4 t_1^2 t_2^2 c_2^2 k_1^2 k_3^2 - \\
 & 6 t_1 t_2^3 c_2 c_3 k_1^3 k_2 - 6 t_1 t_2^3 c_1 c_3 k_1 k_2^3 + 12 t_1 t_2^3 c_1 c_3 k_1 k_2 k_3^2 - 12 t_1 t_2^3 c_1 c_2 k_1 k_2 k_3^2 + \\
 & 12 t_1 t_2^3 c_3^2 k_1^3 k_2 + 2 t_2^4 c_1 c_2 k_1^2 k_2^2 - 4 t_2^4 c_1 c_3 k_1^2 k_2^2 - 4 t_2^4 c_2 c_3 k_1^4 - 4 t_1^4 c_1 c_3 k_2^4 - \\
 & 2 t_1^2 t_2^2 c_2^2 k_1^4 - 2 t_1^2 t_2^2 c_1^2 k_2^4 + 4 t_1^4 c_2^2 k_3^4 - 4 t_1^4 c_1 c_2 k_2^2 k_3^2 + 4 t_1^4 c_2^2 k_1^2 k_3^2 + 2 t_1^4 k_1^2 k_2^2 c_2 c_1 - \\
 & 4 t_1^4 c_2 c_3 k_1^2 k_2^2 + 8 t_1^4 c_2 c_3 k_2^2 k_3^2 - 12 t_1^3 t_2 c_3^2 k_1 k_2^3 + 6 t_1^3 t_2 c_1 c_3 k_1 k_2^3 + 12 t_1^3 t_2 c_1 c_2 k_1 k_2 k_3^2 - \\
 & 12 t_1^3 t_2 c_2 c_3 k_1 k_2 k_3^2 + 6 t_1^3 t_2 c_2 c_3 k_1^3 k_2 - 4 t_1^2 t_2^2 c_1 c_2 k_1^2 k_2^2 - 4 t_2^4 c_1 c_2 k_1^2 k_3^2 + 8 t_2^4 c_1 c_3 k_1^2 k_3^2 + \\
 & 4 c_1^2 t_2^4 k_3^4 + 4 t_2^4 c_1^2 k_2^2 k_3^2 + t_1^4 k_2^4 c_1^2 + 4 t_1^4 c_3^2 k_2^4 + t_1^4 k_1^4 c_2^2) \cdot (-48 d^2 t_1^6 t_2^2 k_2^6 + 64 d^2 t_1^2 t_2^8 k_1^6 - \\
 & 72 d^2 t_1^4 t_2^4 k_1^6 - 128 d^4 t_1^8 t_2^2 k_2^6 + 64 d^4 t_1^6 t_2^4 k_2^6 + d^4 t_1^4 t_2^6 k_1^6 - 2 d^4 t_1^2 t_2^8 k_1^6 + 25 t_1^6 k_2^4 t_2^4 k_3^2 + 4 k_2^6 t_1^{10} - \\
 & 78 t_1^5 t_2^5 k_1 k_2^5 - 154 t_1^7 t_2^3 k_1 k_2^5 + d^4 k_1^6 t_2^{10} + 64 d^4 k_2^6 t_1^{10} + 8 d^2 k_1^6 t_2^{10} + 32 d^2 k_2^6 t_1^{10} + 136 d^2 t_1^5 t_2^5 k_1 k_2^5 + \\
 & 88 d^2 t_1^7 t_2^3 k_1 k_2^5 + 320 d^2 t_1^8 t_2^2 k_1^2 k_2^4 + 440 d^2 t_1^4 t_2^6 k_1^4 k_2^2 - 100 d^2 t_1^6 k_2^2 k_3^2 t_2^4 k_1^2 - 170 d^2 t_1^3 t_2^7 k_1^3 k_2^3 - \\
 & 110 d^2 t_1^5 t_2^5 k_1^3 k_2^3 + 280 d^2 t_1^7 t_2^3 k_1^3 k_2^3 + 1200 d^2 t_1^7 k_2^3 k_3^2 t_2^3 k_1 - 400 d^2 t_1^8 k_2^4 k_3^2 t_2^2 + 300 d^2 t_1^5 k_2^3 k_3^2 t_2^5 k_1 - \\
 & 1200 d^2 t_1^5 k_2 k_3^2 t_2^5 k_1^3 - 25 d^2 t_1^4 k_2^2 k_3^2 t_2^6 k_1^2 - 400 d^2 t_1^4 k_3^2 t_2^6 k_1^4 - 100 d^2 t_1^2 k_3^2 t_2^8 k_1^4 - 370 d^2 t_1^6 t_2^4 k_1^4 k_2^2 - \\
 & 344 d^2 t_1^5 t_2^5 k_1^5 k_2 + 16 d^2 t_1 t_2^9 k_1^5 k_2 + 328 d^2 t_1^3 t_2^7 k_1^5 k_2 - 300 d^2 t_1^3 k_2 k_3^2 t_2^7 k_1^3 - 70 d^2 t_1^2 t_2^8 k_1^4 k_2^2 + \\
 & 465 t_1^8 t_2^2 k_1^2 k_2^2 - 120 d^4 t_1^6 t_2^4 k_1^4 k_2^2 + 160 d^4 t_1^3 t_2^7 k_1^3 k_2^2 - 24 d^4 t_1^3 t_2^7 k_1^5 k_2 + 60 d^4 t_1^2 t_2^8 k_1^4 k_2^2 + \\
 & 12 d^4 t_1 t_2^9 k_1^5 k_2 + 2480 t_1^4 t_2^6 k_1^4 k_2^2 + 2400 t_1^6 k_2^2 k_3^2 t_2^4 k_1^2 - 440 t_1^3 t_2^7 k_1^3 k_2^3 - 1920 t_1^5 t_2^5 k_1^3 k_2^3 - \\
 & 1640 t_1^3 t_2^3 k_1^3 k_2^3 - 800 t_1^7 k_2^2 k_3^2 t_2^3 k_1 + 100 t_1^8 k_2^2 k_3^2 t_2^2 + 16 d^2 t_1^2 t_2^6 k_2^6 - 200 t_1^5 k_2^3 k_3^2 t_2^5 k_1 - \\
 & 3200 t_1^5 k_2 k_3^2 t_2^5 k_1^3 + 600 t_1^4 k_2^2 k_3^2 t_2^6 k_1^2 + 1600 t_1^4 k_3^2 t_2^6 k_1^4 + 400 t_1^2 k_3^2 t_2^8 k_1^4 + 3160 t_1^6 t_2^4 k_1^4 k_2^2 - \\
 & 3168 t_1^5 t_2^5 k_1^5 k_2 - 128 t_1 t_2^9 k_1^5 k_2 - 1504 t_1^3 t_2^7 k_1^5 k_2 - 800 t_1^3 k_2 k_3^2 t_2^7 k_1^3 + 12 t_1^8 t_2^2 k_2^6 + 9 t_1^6 t_2^4 k_2^6 + \\
 & 360 t_1^2 t_2^8 k_1^4 k_2^2 - 224 d^2 t_1^9 t_2 k_1 k_2^5 + 20 d^2 t_1^4 t_2^6 k_1^4 k_2^2 - 340 d^2 t_1^6 t_2^4 k_1^2 k_2^4 + 192 d^4 t_1^9 t_2 k_1 k_2^5 +
 \end{aligned}$$

$$240 d^4 t_1^8 t_2^2 k_1^2 k_2^4 - 384 d^4 t_1^7 t_2^3 k_1 k_2^5 + 160 d^4 t_1^7 t_2^3 k_1^3 k_2^3 - 480 d^4 t_1^6 t_2^4 k_1^2 k_2^4 + 60 d^4 t_1^6 t_2^4 k_1^4 k_2^2 + 192 d^4 t_1^5 t_2^5 k_1 k_2^5 - 320 d^4 t_1^5 t_2^5 k_1^3 k_2^3 + 12 d^4 t_1^5 t_2^5 k_1^5 k_2 + 240 d^4 t_1^4 t_2^6 k_1^2 k_2^4 + 16 k_1^6 t_2^{10} + 288 t_1^2 t_2^8 k_1^6 + 1296 t_1^4 t_2^6 k_1^6 - 68 t_1^9 t_2 k_1 k_2^5 - 100 d^2 t_1^6 k_2^4 t_2^4 k_3^3 + 265 t_1^4 t_2^6 k_1^2 k_2^4 + 770 t_1^6 t_2^4 k_1^2 k_2^4).$$

From this expression it is easy to check that $\text{PP}_{(d, \bar{k})}(\text{Con}_{\bar{c}}(\text{Res}_{t_0}(T_0(\bar{k}, \bar{t}_h), T(\bar{c}, d, \bar{k}, \bar{t}_h))))$ is the last factor in the above expression, and so:

$$\deg_{\bar{t}}(\text{PP}_{(d, \bar{k})}(\text{Con}_{\bar{c}}(\text{Res}_{t_0}(T_0(\bar{k}, \bar{t}_h), T(\bar{c}, d, \bar{k}, \bar{t}_h)))) = 10.$$

Using Theorem 3.22 one has that the total offset degree is $\delta = 10$. In fact, in this case, using elimination techniques, it is possible to check this result, computing the generic offset polynomial:

$$\begin{aligned} g(d, \bar{x}) = & -256x_1^{10} - 640x_1^8x_2^2 - 256x_1^8x_3^2 + 1408x_1^8d^2 - 400x_1^6x_2^4 - 384x_1^6x_2^2x_3^2 + 3232x_1^6x_2^2d^2 + \\ & 1152x_1^6x_3^2d^2 - 3088x_1^6d^4 + 80x_1^4x_2^6 - 16x_1^4x_2^4x_3^2 + 2448x_1^4x_2^4d^2 + 1696x_1^4x_2^2x_3^2d^2 - 5904x_1^4x_2^2d^4 - \\ & 1936x_1^4x_3^2d^4 + 3376x_1^4d^6 + 80x_1^2x_2^8 + 96x_1^2x_2^6x_3^2 + 832x_1^2x_2^6d^2 + 736x_1^2x_2^4x_3^2d^2 - 3744x_1^2x_2^4d^4 - \\ & 2272x_1^2x_2^2x_3^2d^4 + 4672x_1^2x_2^2d^6 + 1440x_1^2x_3^2d^6 - 1840x_1^2d^8 - 16x_2^{10} - 16x_2^8x_3^2 + 208x_2^8d^2 + 192x_2^6x_3^2d^2 - \\ & 928x_2^6d^4 - 736x_2^4x_3^2d^4 + 1696x_2^4d^6 + 960x_2^2x_3^2d^6 - 1360x_2^2d^8 - 400x_3^2d^8 + 400d^{10} + 3200x_1^8x_3 - \\ & 320x_1^6x_2^2x_3 + 3072x_1^6x_3^3 - 12608x_1^6x_3d^2 - 1560x_1^4x_2^4x_3 - 2944x_1^4x_2^2x_3^3 + 496x_1^4x_2^2x_3d^2 - 9088x_1^4x_3^3d^2 + \\ & 18088x_1^4x_3d^4 + 1640x_1^2x_2^6x_3 + 1696x_1^2x_2^4x_3^3 + 7016x_1^2x_2^4x_3d^2 + 5184x_1^2x_2^2x_3^3d^2 - 2184x_1^2x_2^2x_3d^4 + \\ & 8480x_1^2x_3^3d^4 - 11080x_1^2x_3d^6 - 320x_2^8x_3 - 288x_2^6x_3^3 + 2912x_2^6x_3d^2 + 2272x_2^4x_3^3d^2 - 7072x_2^4x_3d^4 - \\ & 3680x_2^2x_3^3d^4 + 2080x_2^2x_3d^6 - 2400x_3^3d^6 + 2400x_3d^8 + 2544x_1^8 - 9144x_1^6x_2^2 - 10752x_1^6x_3^2 - 10520x_1^6d^2 + \\ & 4479x_1^4x_2^4 - 6976x_1^4x_2^2x_3^2 + 25770x_1^4x_2^2d^2 - 11776x_1^4x_3^4 + 21568x_1^4x_3^2d^2 + 16583x_1^4d^4 - 684x_1^2x_2^6 + \\ & 10304x_1^2x_2^4x_3^2 + 2700x_1^2x_2^4d^2 + 9088x_1^2x_2^2x_3^4 + 25056x_1^2x_2^2x_3^2d^2 - 23444x_1^2x_2^2d^4 + 14720x_1^2x_3^4d^2 - \\ & 7840x_1^2x_3^2d^4 - 11980x_1^2d^6 + 24x_2^8 - 2472x_2^6x_3^2 + 160x_2^6d^2 - 1936x_2^4x_3^4 + 14488x_2^4x_3^2d^2 - 2752x_2^4d^4 + \\ & 8480x_2^2x_3^4d^2 - 15160x_2^2x_3^2d^4 + 6080x_2^2d^6 - 400x_3^4d^4 - 3000x_3^2d^6 + 3400d^8 - 19008x_1^6x_3 + \\ & 25896x_1^4x_2^2x_3 + 3328x_1^4x_3^3 + 44072x_1^4x_3d^2 - 6534x_1^2x_2^4x_3 + 23616x_1^2x_2^2x_3^3 + 2484x_1^2x_2^2x_3d^2 + \\ & 15360x_1^2x_3^5 + 15680x_1^2x_3^3d^2 - 31790x_1^2x_3d^4 + 312x_2^6x_3 - 9112x_2^4x_3^3 + 3112x_2^4x_3d^2 - 5760x_2^2x_3^5 + \\ & 31120x_2^2x_3^3d^2 - 22360x_2^2x_3d^4 + 9600x_3^5d^2 - 17400x_3^3d^4 + 7800x_3d^6 - 6240x_1^6 + 3360x_1^4x_2^2 + \\ & 29984x_1^4x_3^2 + 15816x_1^4d^2 - 510x_1^2x_2^4 - 18472x_1^2x_2^2x_3^2 + 11022x_1^2x_2^2d^2 + 14080x_1^2x_3^4 + 8440x_1^2x_3^2d^2 - \\ & 16520x_1^2d^4 + 15x_2^6 + 1319x_2^4x_3^2 - 669x_2^4d^2 - 15680x_2^2x_3^4 + 15010x_2^2x_3^2d^2 + 1045x_2^2d^4 - 6400x_3^6 + \\ & 27200x_3^4d^2 - 28825x_3^2d^4 + 8025d^6 + 14880x_1^4x_3 - 4800x_1^2x_2^2x_3 - 13120x_1^2x_3^3 + 14320x_1^2x_3d^2 + \\ & 270x_2^4x_3 + 1720x_2^2x_3^3 - 4270x_2^2x_3d^2 - 9600x_3^5 + 17400x_3^3d^2 - 7800x_3d^4 - 400x_1^4 + 200x_1^2x_2^2 - \\ & 11040x_1^2x_3^2 + 7840x_1^2d^2 - 25x_2^4 + 1440x_2^2x_3^2 - 140x_2^2d^2 - 400x_3^4 - 3000x_3^2d^2 + 3400d^4 + 800x_1^2x_3 - \\ & 200x_2^2x_3 + 2400x_3^3 - 2400x_3d^2 - 400x_3^2 + 400d^2 \end{aligned}$$

This is, as predicted by our formula, a polynomial of degree 10 in \bar{x} .

Example 3.24. To illustrate the behavior of the degree formula in the case of non-proper parametrizations, let us consider the surface Σ defined by the parametrization:

$$P(t_1, t_2) = (t_1^3, t_2, t_1^6 + t_2^2).$$

This is a parametrization of the circular paraboloid with implicit equation $y_3 = y_1^2 + y_2^2$; the parametrization P has been obtained by replacing t_1 with t_1^3 in the usual proper parametrization \tilde{P} of Σ , which is given by:

$$\tilde{P}(t_1, t_2) = (t_1, t_2, t_1^2 + t_2^2).$$

Thus, the tracing index of the parametrization P in this example is $\mu = 3$. Computing with P we obtain the following associated normal vector:

$$N(\bar{t}_h) = (-2t_1^3, -2t_0^2 t_2, t_0^3).$$

Then the auxiliary curves are:

$$T_0(\bar{t}_h) = -k_2 t_1^3 + k_1 t_0^2 t_2$$

$$T_1(\bar{t}_h) = 4t_1^{18} k_2^2 + 12t_1^{12} k_2^2 t_2^2 t_0^4 - 8t_1^{12} k_2 k_3 t_2 t_0^5 + 12t_1^6 k_2^2 t_2^4 t_0^8 - 16t_1^6 k_2 t_2^3 t_0^9 k_3 + 4t_1^6 k_3^2 t_2^2 t_0^{10} + 4t_2^6 t_0^{12} k_2^2 - 8t_2^5 t_0^{13} k_2 k_3 + 4t_2^4 t_0^{14} k_3^2 + t_0^6 k_2^2 t_1^{12} + 2t_0^{10} k_2^2 t_1^6 t_2^2 - 2t_0^{11} k_2 t_1^6 k_3 t_2 + t_0^{14} k_2^2 t_2^4 - 2t_0^{15} k_2 t_2^3 k_3 + t_0^{16} k_3^2 t_2^2 - d^2 t_0^{18} k_2^2 - 4d^2 t_0^{17} k_2 k_3 t_2 - 4d^2 t_0^{16} k_3^2 t_2^2$$

$$T_2(\bar{t}_h) = 4t_1^{12} k_3^2 t_0^6 - 8t_1^{15} k_3 t_0^3 k_1 - 16t_1^9 k_3 t_0^7 k_1 t_2^2 + 4t_1^{18} k_1^2 + 12t_1^{12} k_1^2 t_2^2 t_0^4 + 12t_1^6 k_1^2 t_2^4 t_0^8 + 4t_1^6 k_3^2 t_2^2 t_0^{10} - 8t_2^4 t_0^{11} k_3 t_1^3 k_1 + 4t_2^6 t_0^{12} k_1^2 + t_0^{12} k_3^2 t_1^6 - 2t_0^9 k_3 t_1^9 k_1 - 2t_0^{13} k_3 t_1^3 k_1 t_2^2 + t_0^6 k_1^2 t_1^{12} + 2t_0^{10} k_1^2 t_1^6 t_2^2 + t_0^{14} k_1^2 t_2^4 - 4d^2 t_0^{12} t_1^6 k_3^2 - 4d^2 t_0^{15} k_3 t_1^3 k_1 - d^2 t_0^{18} k_1^2$$

$$T_3(\bar{t}_h) = t_0^6 (k_2 t_1^3 - k_1 t_0^2 t_2)^2 (4t_1^6 + t_0^6 + 4t_2^2 t_0^4 - 4d^2 t_0^6)$$

Denoting, as in the degree formula,

$$R(\bar{c}, d, \bar{k}, \bar{t}) = \text{Res}_{t_0} (T_0(\bar{k}, \bar{t}_h), T(\bar{c}, d, \bar{k}, \bar{t}_h)),$$

one has:

$$\begin{aligned} R(\bar{c}, d, \bar{k}, \bar{t}) = & (k_1^2 a_2 + k_2^2 a_1)^2 t_1^{36} (-8k_2^{12} t_1^{12} k_1^2 t_2^6 k_3^4 d^2 + 16k_1^{18} t_2^{18} + 16k_2^{12} k_1^6 t_2^{18} + \\ & 96k_2^{10} k_1^8 t_2^{18} + 240k_2^8 k_1^{10} t_2^{18} + 320k_2^6 k_1^{12} t_2^{18} + 240k_1^{14} t_2^{18} k_2^4 + 96k_1^{16} t_2^{18} k_2^2 + t_1^{18} d^4 k_2^{18} + \\ & t_1^6 k_1^4 t_2^{12} k_2^{14} + 8t_1^3 k_1^5 t_2^{15} k_2^{13} + 4t_1^6 k_1^6 t_2^{12} k_2^{12} + 40t_1^3 k_1^7 t_2^{15} k_2^{11} + 6t_1^6 k_1^8 t_2^{12} k_2^{10} + 80t_1^3 k_1^9 t_2^{15} k_2^9 + \\ & 80t_1^3 k_1^{11} t_2^{15} k_2^7 + 4k_2^8 t_1^6 k_1^{10} t_2^{12} + 40k_2^5 t_1^3 k_1^{13} t_2^{15} + k_2^6 t_1^6 k_1^{12} t_2^{12} + 8k_2^3 t_1^3 k_1^{15} t_2^{15} - 32k_2^6 t_1^6 k_1^{10} t_2^{12} d^2 k_3^2 + \\ & 16k_2^{10} t_1^6 k_1^4 t_2^{12} k_3^4 + 32k_2^8 t_1^6 k_1^6 t_2^{12} k_3^4 - 192k_2^7 t_1^3 k_1^{15} t_2^{12} k_3^2 - 128k_2^5 t_1^3 k_1^{11} t_2^{15} k_3^2 + 16k_2^6 t_1^6 k_1^8 t_2^{12} k_3^4 - \\ & 32k_2^3 t_1^3 k_1^{13} t_2^{15} k_3^2 - 4k_2^{11} t_1^9 k_1^5 t_2^9 k_3^2 + 8k_2^{11} t_1^9 k_1^3 t_2^9 k_3^4 - 2k_2^9 t_1^9 k_1^7 t_2^9 k_3^2 + 8k_2^9 t_1^9 k_1^5 t_2^9 k_3^4 - \\ & 48k_2^8 t_1^6 k_1^8 t_2^{12} k_3^2 - 16k_2^6 t_1^6 k_1^{10} t_2^{12} k_3^2 - 32k_2^{12} k_1^4 t_2^{12} t_1^6 d^2 k_3^2 - 32k_2^{11} k_1^5 t_2^{15} t_1^3 k_3^2 - 128k_2^9 k_1^7 t_2^{15} t_1^3 k_3^2 + \\ & 16k_2^{12} t_1^{12} k_1^2 t_2^6 d^4 k_3^4 - 144k_2^{11} t_1^9 k_1^5 t_2^9 d^2 k_3^2 - 32k_2^{11} t_1^9 k_1^3 t_2^9 d^2 k_3^4 - 96k_2^{10} t_1^6 k_1^6 t_2^{12} d^2 k_3^2 - \\ & 2t_1^{12} d^2 k_2^{16} k_1^2 t_2^6 - 2t_1^{15} d^2 k_2^{15} k_1 t_2^3 k_3^2 - 8t_1^9 d^2 k_2^{15} k_1^3 t_2^9 - 8t_1^{15} d^4 k_2^{15} k_1 t_2^3 k_3^2 - 4t_1^{12} d^2 k_2^{14} k_1^4 t_2^6 - \\ & 24t_1^{12} d^2 k_2^{14} k_1^2 t_2^6 k_3^2 - 24t_1^9 d^2 k_2^{13} k_1^5 t_2^9 - 2t_1^{12} d^2 k_2^{12} k_1^6 t_2^6 - 24t_1^{12} d^2 k_2^{12} k_1^4 t_2^6 k_3^2 - 24t_1^9 d^2 k_2^{11} k_1^7 t_2^9 - \\ & 8t_1^9 d^2 k_2^9 k_1^9 t_2^9 - 2t_1^9 k_1^3 t_2^9 k_2^{13} k_3^2 - 72t_1^9 k_1^3 t_2^9 k_2^{13} d^2 k_3^2 - 16t_1^6 k_1^4 t_2^{12} k_2^{12} k_3^2 - 48t_1^6 k_1^6 t_2^{12} k_2^{10} k_3^2 + \\ & k_2^{12} t_1^{12} k_1^2 t_2^6 k_3^4 - 72k_2^9 t_1^9 k_1^7 t_2^9 d^2 k_3^2 - 32k_2^9 t_1^9 k_1^5 t_2^9 d^2 k_3^4 - 96k_2^8 t_1^6 k_1^8 t_2^{12} d^2 k_3^2). \end{aligned}$$

From this expression it is easy to check that

$$\deg_{\bar{t}} (\text{PP}_{(d, \bar{k})} (\text{Con}_{\bar{c}} (\text{Res}_{t_0} (T_0(\bar{k}, \bar{t}_h), T(\bar{c}, d, \bar{k}, \bar{t}_h)))) = 18.$$

This agrees with the expected result $\mu \cdot \delta = 3 \cdot 6 = 18$.

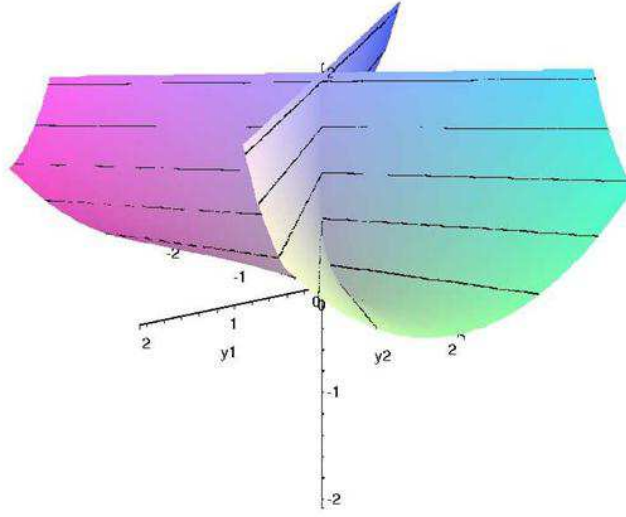


Figure 4: The Whitney Umbrella

Example 3.25. Let Σ be the surface (Whitney Umbrella) with implicit equation $y_1^2 - y_2^2 y_3 = 0$. A proper rational parametrization of Σ is given by:

$$P(t_1, t_2) = (t_1 t_2, t_2, t_1).$$

This surface is illustrated in Figure 4. The homogeneous associated normal vector is

$$N(\bar{t}_h) = (2t_1 t_2, -2t_1^2, -t_0 t_2).$$

Then the auxiliary curves are:

$$T_0(\bar{t}_h) = -k_1 t_0^2 t_2^2 + 2 k_1 t_1^4 + t_1 t_0^2 k_2 t_2 - 2 t_1^3 t_0 k_3 + 2 t_1^3 t_2 k_2 - 2 t_1 t_2^2 k_3 t_0$$

$$T_1(\bar{t}_h) = 4 t_1^6 t_2^2 k_2^2 - 8 t_1^4 t_2^3 k_2 k_3 t_0 + 4 t_1^2 t_2^4 k_3^2 t_0^2 + 4 t_1^8 k_2^2 - 8 t_1^6 k_2 k_3 t_0 t_2 + 4 t_1^4 k_3^2 t_0^2 t_2^2 + t_0^2 t_2^2 k_2^2 t_1^4 - 2 t_0^3 t_2^3 k_2 t_1^3 k_3 + t_0^4 t_2^4 k_3^2 - d^2 t_2^6 k_2^2 t_0^2 + 4 d^2 t_2^5 k_2 t_1^2 k_3 t_0 - 4 d^2 t_2^4 k_3^2 t_1^4$$

$$T_2(\bar{t}_h) = 4 t_1^4 k_3^2 t_0^2 t_2^2 - 8 t_1^5 t_2^2 k_3 t_0 k_1 + 4 t_1^6 t_2^2 k_1^2 + 4 t_1^6 k_3^2 t_0^2 - 8 t_1^7 k_3 t_0 k_1 + 4 t_1^8 k_1^2 + t_0^4 t_2^2 k_3^2 t_1^2 - 2 t_0^3 t_2^3 k_3 t_1^3 k_1 + t_0^2 t_2^4 k_1^2 t_1^4 - 4 d^2 t_2^6 t_1^3 k_3^2 - 4 d^2 t_2^6 t_1 k_3 k_1 t_0 - d^2 t_2^6 k_1^2 t_0^2$$

$$T_3(\bar{t}_h) = 4 t_0^2 t_2^2 k_2^2 t_1^4 - 8 t_1^3 t_2^3 k_2 t_0^2 k_1 + 4 t_1^2 t_2^4 k_1^2 t_0^2 + 4 t_1^6 k_2^2 t_0^2 - 8 t_1^5 k_2 t_0^2 k_1 t_2 + 4 t_0^2 t_2^2 k_1^2 t_1^4 + t_0^4 t_2^2 k_2^2 t_1^2 - 2 t_0^4 t_2^3 k_2 t_1 k_1 + t_0^4 t_2^4 k_1^2 - 4 d^2 t_2^6 k_2^2 t_1^2 - 8 d^2 t_2^5 t_1^3 k_2 k_1 - 4 d^2 t_2^4 k_1^2 t_1^4$$

Denoting, as in the degree formula,

$$R(\bar{c}, d, \bar{k}, \bar{t}) = \text{Res}_{t_0} (T_0(\bar{k}, \bar{t}_h), T(\bar{c}, d, \bar{k}, \bar{t}_h)),$$

one has:

$$\begin{aligned}
R(\bar{c}, d, \bar{k}, \bar{t}) = & 4t_2^2 t_1^4 (-28 t_1^6 t_2^8 k_3^2 d^2 k_1^2 + 4k_2^4 t_1^8 t_2^6 d^2 + k_1^4 d^4 t_2^{12} t_1^2 - 6d^2 k_1^4 t_2^8 t_1^6 + k_2^2 k_1^2 d^4 t_2^{14} - \\
& 4k_1^4 d^2 t_2^{10} t_1^4 + 12k_1^4 t_2^6 t_1^8 - 20t_1^4 t_2^{10} d^2 k_1^2 k_3^2 - 4k_2^2 k_1^2 d^4 t_2^{12} t_1^2 - 4k_2^2 t_1^8 t_2^6 k_3^2 d^2 - 24k_2 t_1^{11} t_2^3 k_1 k_3^2 - \\
& 2k_2 d^4 k_1^3 t_2^{11} t_1^3 + 4k_2 k_1^3 d^2 t_2^9 t_1^5 + 2k_2 k_1^3 d^4 t_2^{13} t_1 + 16k_2 t_1^5 t_2^9 d^2 k_3^2 k_1 - 8k_2 t_1^{13} t_2 k_3^2 k_1 - 8k_2 t_1^7 t_2^7 k_3^2 k_1 + \\
& 32k_2 t_1^7 t_2^7 k_3^2 d^2 k_1 + 16k_2 d^2 k_1^3 t_2^7 t_1^7 + 16k_2 k_3^2 k_1 d^2 t_2^9 t_1^9 - 24k_2 k_3^2 k_1 t_2^9 t_1^9 - 12t_1^8 t_2^6 k_3^2 d^2 k_1^2 - \\
& 4t_1^{10} t_2^4 k_3^4 d^2 + 4t_1^{12} t_2^2 k_1^2 k_3^2 + 4t_1^6 t_2^8 k_3^2 k_1^2 - 16t_1^8 t_2^6 k_3^4 d^2 - 4t_1^2 t_2^{12} d^2 k_3^4 + 12t_1^8 t_2^6 k_3^2 k_1^2 + 12t_1^{10} t_2^4 k_3^2 k_1^2 - \\
& 6k_3^3 k_1 t_2^5 t_1^9 + 12t_1^{10} t_2^4 k_3^2 k_2^2 + 4t_1^8 t_2^6 k_3^2 k_2^2 - 16t_1^4 t_2^{10} k_3^4 d^2 + 4t_1^4 t_2^{10} k_3^2 d^2 k_2^2 - 24t_1^6 t_2^8 k_3^4 d^2 + t_1^{12} t_2^{12} d^4 k_2^2 + \\
& 2t_1^6 t_2^8 d^2 k_2^2 + 4t_1^6 t_2^8 k_3^2 d^2 k_2^2 - 20k_2^3 k_1 t_2 t_1^{13} + 4k_2^4 t_1^{14} + 9k_1^4 t_2^4 t_1^{10} - 22k_2^3 k_1 t_2^3 t_1^{11} + 37k_2^2 k_1^2 t_2^2 t_1^{12} + \\
& 44k_2^2 k_1^2 t_2^4 t_1^{10} + 10k_2^2 k_1^2 d^2 t_2^{10} t_1^4 + 13k_2^2 k_1^2 t_2^6 t_1^8 - 12k_2 k_1^3 t_2^7 t_1^7 - 30k_2 k_1^3 t_2^3 t_1^{11} - 38k_2 k_1^3 t_2^5 t_1^9 - \\
& 4k_2 d^2 k_1^3 t_2^{11} t_1^3 + t_1^{10} t_2^4 k_2^4 - 4t_1^2 t_2^{12} d^2 k_1^2 k_3^2 + 4k_1^4 t_2^8 t_1^6 + 4k_2^2 t_1^{14} k_3^2 + 4k_2^4 t_1^{12} t_2^2 + 12t_1^{12} t_2^2 k_3^2 k_2^2 - \\
& 8k_2^3 k_1 d^2 t_2^9 t_1^5 - 12k_2^3 k_1 d^2 t_2^7 t_1^7 + 2k_2^3 k_1 d^4 t_2^{11} t_1^3 + 4k_2^3 k_1 d^2 t_2^5 t_1^9 - 2k_2^3 k_1 d^4 t_2^{13} t_1 + 8k_2^2 k_1^2 d^2 t_2^8 t_1^6 - \\
& 14k_2^2 k_1^2 d^2 t_2^6 t_1^8 - 4k_2^2 t_1^{10} d^2 t_2^2 k_3^2 + k_2^2 k_1^2 d^4 t_2^{10} t_1^4) \cdot (8c_3 c_1 k_1 k_2^2 k_2 t_1 t_2^3 - 4c_3 c_1 k_1 t_1^3 k_2^2 t_2 - \\
& 8c_3 c_1 k_1 t_1^3 k_3^2 k_2 t_2 + 4c_3 c_1 k_3^2 k_2^2 t_2^4 - 4c_3 c_1 k_2^2 t_1^2 t_2^2 + 4c_3 c_1 t_1^4 k_2^2 k_3^2 - 4c_1^2 k_3^2 k_2^2 t_1^2 t_2^2 + 8c_3^2 k_1^3 t_1 k_2 t_2^3 - \\
& 8c_3^2 k_1^3 t_1^3 t_2 k_2 + 4c_3^2 k_2^2 k_1^2 t_2^4 + 4c_3^2 k_1^2 t_1^4 k_2^2 - 16c_3^2 k_1^2 t_2^2 k_2^2 t_1^2 - 8c_3^2 k_1 k_2^3 t_1 t_2^3 + 8c_3^2 k_1 t_1^3 k_2^3 t_2 + \\
& 4c_3^2 k_2^4 t_1^2 t_2^2 + c_2^2 t_1^2 t_2^2 k_1^4 + 4c_2^2 t_1^2 t_1^4 k_3^4 + 4c_2 c_1 k_3^4 t_2^4 + 4c_2 c_1 k_3^4 t_1^4 + 4c_3 c_2 t_1^2 t_2^2 k_1^4 + 4c_3 c_2 k_1^3 t_1 k_2 t_2^3 - \\
& 4c_3 c_2 k_1^3 t_1^3 t_2 k_2 + 4c_3 c_2 k_1^2 t_2^4 k_3^2 - 4c_3 c_2 k_1^2 t_2^2 k_2^2 t_1^2 + 4c_3 c_2 k_1^2 t_1^4 k_3^2 - 8c_3 c_2 k_1 k_2^3 k_2 t_1 t_2^3 + \\
& 8c_3 c_2 k_1 t_1^3 k_2^3 k_2 t_2 + 8c_3 c_2 k_3^2 k_2^2 t_1^2 t_2^2 + 4c_2^2 k_1^2 t_1^2 k_3^2 t_2^2 + 4c_2 c_1 k_1^2 t_1^2 k_3^2 t_2^2 + 2c_2 c_1 k_1^2 t_2^2 k_2^2 t_1^2 + \\
& 8c_2 c_1 k_1 k_2^3 k_2 t_1 t_2^3 - 8c_2 c_1 k_1 t_1^3 k_2^3 k_2 t_2 - 4c_2 c_1 k_3^2 k_2^2 t_1^2 t_2^2 + c_1^2 k_2^4 t_1^2 t_2^2 + 4c_1^2 t_2^2 t_1^2 k_3^4 + 4c_3^2 t_1^2 t_2^2 k_1^4 + \\
& 8c_3 c_1 k_1^2 t_2^2 k_3^2 t_2^2 + 4c_3 c_1 k_1^2 t_2^2 k_2^2 t_1^2 + 4c_3 c_1 k_1 k_2^3 t_1 t_2^3).
\end{aligned}$$

From this expression it is easy to check that

$$\deg_{\bar{t}}(\text{PP}_{(d, \bar{k})}(\text{Con}_{\bar{c}}(\text{Res}_{t_0}(T_0(\bar{k}, \bar{t}_h), T(\bar{c}, d, \bar{k}, \bar{t}_h)))) = 14,$$

and then, using Theorem 3.22 one concludes that the total offset degree in \bar{x} is $\delta = 14$.

In fact, in this case, using elimination techniques, it is possible to check this result, computing the generic offset polynomial (see Appendix 3.4, page 84). This is indeed a polynomial of degree 14 in \bar{x} .

Appendix: Computational Complements

Coefficients $c_j^{(i)}$ in Lemma 2.10 (page 40).

The polynomials s_i can be expressed as follows:

$$s_i = c_1^{(i)} b^P + c_2^{(i)} \text{nor}_{(1,2)}^P + c_3^{(i)} \text{nor}_{(1,3)}^P + c_4^{(i)} \text{nor}_{(2,3)}^P + c_5^{(i)} w^P + c_6^{(i)} \ell_1 + c_7^{(i)} \ell_2 + c_8^{(i)} \ell_3$$

where $c_j^{(i)} \in \mathbb{C}[d, \bar{k}, l, r, \bar{t}, \bar{x}]$ for $i = 0, \dots, 3$, $j = 1, \dots, 8$ are the following polynomials:

$$c_1^{(0)} = 0$$

$$c_2^{(0)} = k_3$$

$$c_3^{(0)} = -k_2$$

$$c_4^{(0)} = k_1$$

$$c_5^{(0)} = P_0 n_2 k_3 - P_0 n_3 k_2$$

$$c_6^{(0)} = -P_0 n_1 k_3 + P_0 n_3 k_1$$

$$c_7^{(0)} = P_0 n_1 k_2 - P_0 n_2 k_1$$

$$c_8^{(0)} = 0$$

$$c_1^{(1)} = -2 P_0 n_1^3 n_2 k_1 k_2 r + P_0 n_1^2 n_2^2 k_1^2 r - 2 P_0 n_1 n_2^3 k_1 k_2 r - 2 P_0 n_1 n_2 n_3^2 k_1 k_2 r + P_0 n_2^4 k_1^2 r + P_0 n_2^2 n_3^2 k_1^2 r + n_1^2 k_2^2$$

$$c_2^{(1)} = 2 P_1 P_0 n_1^3 k_1 k_2 r - P_1 P_0 n_1^2 n_2 k_1^2 r + 2 P_1 P_0 n_1 n_2^2 k_1 k_2 r - P_1 P_0 n_2^3 k_1^2 r + P_1 n_2 k_2^2 - P_2 P_0 n_1^3 k_1^2 r - P_2 P_0 n_1 n_2^2 k_1^2 r + P_2 n_1 k_2^2 - 2 P_2 n_2 k_1 k_2 + 2 P_3 P_0 n_1^2 n_3 k_1 k_2 r + P_0^2 n_1^3 k_1^2 r x_2 - 2 P_0^2 n_1^3 k_1 k_2 r x_1 + P_0^2 n_1^2 n_2 k_1^2 r x_1 - 2 P_0^2 n_1^2 n_3 k_1 k_2 r x_3 + P_0^2 n_1 n_2^2 k_1^2 r x_2 - 2 P_0^2 n_1 n_2^2 k_1 k_2 r x_1 + P_0^2 n_2^3 k_1^2 r x_1 - P_0 n_1 k_2^2 x_2 + P_0 n_2 k_2^2 x_1$$

$$c_3^{(1)} = 2 P_1 P_0 n_1 n_2 n_3 k_1 k_2 r - P_1 P_0 n_2^2 n_3 k_1^2 r + P_1 n_3 k_2^2 - 2 P_2 P_0 n_1^2 n_3 k_1 k_2 r - P_3 P_0 n_1 n_2^2 k_1^2 r + P_3 n_1 k_2^2 - 2 P_3 n_2 k_1 k_2 + 2 P_0^2 n_1^2 n_3 k_1 k_2 r x_2 + P_0^2 n_1 n_2^2 k_1^2 r x_3 - 2 P_0^2 n_1 n_2 n_3 k_1 k_2 r x_1 + P_0^2 n_2^2 n_3 k_1^2 r x_1 - P_0 n_1 k_2 k_3 x_2 + 2 P_0 n_2 k_2 k_3 x_1 - P_0 n_3 k_2^2 x_1$$

$$c_4^{(1)} = -2 P_1 n_3 k_1 k_2 + P_2 P_0 n_1^2 n_3 k_1^2 r + P_2 n_3 k_1^2 + P_3 P_0 n_1^2 n_2 k_1^2 r + P_3 n_2 k_1^2 - P_0^2 n_1^2 n_2 k_1^2 r x_3 - P_0^2 n_1^2 n_3 k_1^2 r x_2 - P_0 n_2 k_1 k_3 x_1 + P_0 n_3 k_1 k_2 x_1$$

$$c_5^{(1)} = 2 P_1 P_0 n_1^2 k_2^2 - 2 P_1 P_0 n_2 n_3 k_2 k_3 + 2 P_1 P_0 n_3^2 k_2^2 - 2 P_2 P_0 n_1^2 k_1 k_2 + 2 P_2 P_0 n_1 n_2 k_2^2 - 2 P_2 P_0 n_2^2 k_1 k_2 + P_2 P_0 n_2 n_3 k_1 k_3 - P_2 P_0 n_3^2 k_1 k_2 + 2 P_3 P_0 n_1 n_2 k_2 k_3 - P_3 P_0 n_2^2 k_1 k_3 - P_3 P_0 n_2 n_3 k_1 k_2 + P_0^2 n_1^2 k_1 k_2 x_2 - P_0^2 n_1^2 k_2^2 x_1 - 2 P_0^2 n_1 n_2 k_2^2 x_2 - 2 P_0^2 n_1 n_2 k_3^2 x_2 + P_0^2 n_2^2 k_1 k_2 x_2 + P_0^2 n_2^2 k_1 k_3 x_3 + P_0^2 n_2^2 k_2^2 x_1 + 2 P_0^2 n_2 n_3 k_2 k_3 x_1 - P_0^2 n_3^2 k_2^2 x_1$$

$$c_6^{(1)} = -2 P_1 P_0 n_1^2 k_1 k_2 + P_1 P_0 n_1 n_3 k_2 k_3 - 2 P_1 P_0 n_3^2 k_1 k_2 + 2 P_2 P_0 n_1^2 k_1^2 - 2 P_2 P_0 n_1 n_2 k_1 k_2 + 2 P_2 P_0 n_2^2 k_1^2 + P_2 P_0 n_3^2 k_1^2 - P_3 P_0 n_1^2 k_2 k_3 + P_3 P_0 n_2 n_3 k_1^2 - P_0^2 n_1^2 k_1^2 x_2 + P_0^2 n_1^2 k_1 k_2 x_1 + P_0^2 n_1^2 k_2 k_3 x_3 + 2 P_0^2 n_1 n_2 k_1 k_2 x_2 - 2 P_0^2 n_1 n_2 k_1 k_3 x_3 + 2 P_0^2 n_1 n_2 k_3^2 x_1 - P_0^2 n_1 n_3 k_2 k_3 x_1 - P_0^2 n_2^2 k_1^2 x_2 - P_0^2 n_2^2 k_1 k_2 x_1 - P_0^2 n_2 n_3 k_1 k_3 x_1 + P_0^2 n_3^2 k_1 k_2 x_1$$

$$c_7^{(1)} = -P_1 P_0 n_1 n_3 k_2^2 + 2 P_1 P_0 n_2 n_3 k_1 k_2 - P_2 P_0 n_2 n_3 k_1^2 + P_3 P_0 n_1^2 k_2^2 - 2 P_3 P_0 n_1 n_2 k_1 k_2 + P_3 P_0 n_2^2 k_1^2 - P_0^2 n_1^2 k_2^2 x_3 + 2 P_0^2 n_1 n_2 k_1 k_2 x_3 + 2 P_0^2 n_1 n_2 k_1 k_3 x_2 - 2 P_0^2 n_1 n_2 k_2 k_3 x_1 + P_0^2 n_1 n_3 k_2^2 x_1 - P_0^2 n_2^2 k_1^2 x_3 - P_0^2 n_2 n_3 k_1 k_2 x_1$$

$$c_8^{(1)} = 2 P_1 P_2 n_1^2 k_1 k_2 - 2 P_1 P_0 n_1^2 k_1 k_2 x_2 - P_2^2 n_1^2 k_1^2 + 2 P_2^2 n_1 n_2 k_1 k_2 - P_2^2 n_2^2 k_1^2 + 2 P_2 P_0 n_1^2 k_1^2 x_2 - 2 P_2 P_0 n_1^2 k_1 k_2 x_1 - 4 P_2 P_0 n_1 n_2 k_1 k_2 x_2 + 2 P_2 P_0 n_2^2 k_1^2 x_2 +$$

$$2 P_3^2 n_1 n_2 k_1 k_2 - P_3^2 n_2^2 k_1^2 - 4 P_3 P_0 n_1 n_2 k_1 k_2 x_3 + 2 P_3 P_0 n_2^2 k_1^2 x_3 - P_0^2 n_1^2 k_1^2 x_2^2 + 2 P_0^2 n_1^2 k_1 k_2 x_1 x_2 - 2 P_0^2 n_1 n_2 k_1 k_2 d^2 + 2 P_0^2 n_1 n_2 k_1 k_2 x_2^2 + 2 P_0^2 n_1 n_2 k_1 k_2 x_3^2 + P_0^2 n_2^2 k_1^2 d^2 - P_0^2 n_2^2 k_1^2 x_2^2 - P_0^2 n_2^2 k_1^2 x_3^2$$

$$c_1^{(2)} = P_0 n_1^2 n_2^2 k_3^2 r - 2 P_0 n_1^2 n_2 n_3 k_2 k_3 r + P_0 n_1^2 n_3^2 k_2^2 r + P_0 n_2^4 k_3^2 r - 2 P_0 n_2^3 n_3 k_2 k_3 r + P_0 n_2^2 n_3^2 k_2^2 r + P_0 n_2^2 n_3^2 k_3^2 r - 2 P_0 n_2 n_3^3 k_2 k_3 r + P_0 n_3^4 k_2^2 r$$

$$c_2^{(2)} = -P_1 P_0 n_1^2 n_2 k_3^2 r + 2 P_1 P_0 n_1^2 n_3 k_2 k_3 r - P_1 P_0 n_2^3 k_3^2 r + 2 P_1 P_0 n_2^2 n_3 k_2 k_3 r - P_2 P_0 n_1^3 k_3^2 r - P_2 P_0 n_1 n_2^2 k_3^2 r + P_0^2 n_1^3 k_3^2 r x_2 + P_0^2 n_1^2 n_2 k_3^2 r x_1 - 2 P_0^2 n_1^2 n_3 k_2 k_3 r x_1 + P_0^2 n_1 n_2^2 k_3^2 r x_2 + P_0^2 n_2^3 k_3^2 r x_1 - 2 P_0^2 n_2^2 n_3 k_2 k_3 r x_1$$

$$c_3^{(2)} = -P_1 P_0 n_1^2 n_3 k_2^2 r - P_1 P_0 n_2^2 n_3 k_2^2 r - P_1 P_0 n_2^2 n_3 k_3^2 r + 2 P_1 P_0 n_2 n_3^2 k_2 k_3 r - P_1 P_0 n_3^3 k_2^2 r + 2 P_2 P_0 n_1^3 k_2 k_3 r + 2 P_2 P_0 n_1 n_2^2 k_2 k_3 r - P_2 P_0 n_1^2 n_3 k_2^2 r - P_3 P_0 n_1^2 k_2^2 r - P_3 P_0 n_1 n_2^2 k_2^2 r - P_3 P_0 n_1 n_2^2 k_3^2 r + 2 P_3 P_0 n_1 n_2 n_3 k_2 k_3 r - P_3 P_0 n_1 n_3^2 k_2^2 r + P_0^2 n_1^3 k_2^2 r x_3 - 2 P_0^2 n_1^3 k_2 k_3 r x_2 + P_0^2 n_1^2 n_3 k_2^2 r x_1 + P_0^2 n_1 n_2^2 k_2^2 r x_3 - 2 P_0^2 n_1 n_2^2 k_2 k_3 r x_2 + P_0^2 n_1 n_2^2 k_3^2 r x_3 - 2 P_0^2 n_1 n_2 n_3 k_2 k_3 r x_3 + P_0^2 n_1 n_3^2 k_2^2 r x_3 + P_0^2 n_2^2 n_3 k_2^2 r x_1 + P_0^2 n_2^2 n_3 k_3^2 r x_1 - 2 P_0^2 n_2 n_3^2 k_2 k_3 r x_1 + P_0^2 n_3^3 k_2^2 r x_1$$

$$c_4^{(2)} = 2 P_2 P_0 n_1^2 n_2 k_2 k_3 r - P_2 P_0 n_1^2 n_3 k_2^2 r + P_2 P_0 n_1^2 n_3 k_3^2 r + 2 P_2 P_0 n_2^3 k_2 k_3 r - P_2 P_0 n_2^2 n_3 k_2^2 r + 2 P_2 P_0 n_2 n_3^2 k_2 k_3 r - P_2 P_0 n_3^3 k_2^2 r + P_2 n_3 k_3^2 - P_3 P_0 n_1^2 n_2 k_2^2 r + P_3 P_0 n_1^2 n_2 k_3^2 r - 2 P_3 P_0 n_1^2 n_3 k_2 k_3 r - P_3 P_0 n_2^2 k_2^2 r - P_3 P_0 n_2 n_3^2 k_2^2 r + P_3 n_2 k_3^2 - 2 P_3 n_3 k_2 k_3 + P_0^2 n_1^2 n_2 k_2^2 r x_3 - 2 P_0^2 n_1^2 n_2 k_2 k_3 r x_2 - P_0^2 n_1^2 n_2 k_3^2 r x_3 + P_0^2 n_1^2 n_3 k_2^2 r x_2 + 2 P_0^2 n_1^2 n_3 k_2 k_3 r x_3 - P_0^2 n_1^2 n_3 k_3^2 r x_2 + P_0^2 n_2^2 k_2^2 r x_3 - 2 P_0^2 n_2^2 k_2 k_3 r x_2 + P_0^2 n_2^2 n_3 k_2^2 r x_2 + P_0^2 n_2 n_3^2 k_2^2 r x_3 - 2 P_0^2 n_2 n_3^2 k_2 k_3 r x_2 + P_0^2 n_3^3 k_2^2 r x_2 - P_0 n_2 k_3^2 x_3 + P_0 n_3 k_3^2 x_2$$

$$c_5^{(2)} = 0$$

$$c_6^{(2)} = 2 P_2 P_0 n_1^2 k_3^2 + 2 P_2 P_0 n_2^2 k_3^2 - 2 P_3 P_0 n_1^2 k_2 k_3 - 2 P_3 P_0 n_2^2 k_2 k_3 + 2 P_3 P_0 n_2 n_3 k_3^2 - 2 P_3 P_0 n_3^2 k_2 k_3 + P_0^2 n_1^2 k_2 k_3 x_3 - P_0^2 n_1^2 k_3^2 x_2 + P_0^2 n_2^2 k_2 k_3 x_3 - P_0^2 n_2^2 k_3^2 x_2 - 2 P_0^2 n_2 n_3 k_3^2 x_3 + P_0^2 n_3^2 k_2 k_3 x_3 + P_0^2 n_3^2 k_3^2 x_2$$

$$c_7^{(2)} = -2 P_2 P_0 n_1^2 k_2 k_3 - 2 P_2 P_0 n_2^2 k_2 k_3 + 2 P_3 P_0 n_1^2 k_2^2 + 2 P_3 P_0 n_2^2 k_2^2 - 2 P_3 P_0 n_2 n_3 k_2 k_3 + 2 P_3 P_0 n_3^2 k_2^2 - P_0^2 n_1^2 k_2^2 x_3 + P_0^2 n_1^2 k_2 k_3 x_2 - P_0^2 n_2^2 k_2^2 x_3 + P_0^2 n_2^2 k_2 k_3 x_2 + 2 P_0^2 n_2 n_3 k_2 k_3 x_3 - P_0^2 n_3^2 k_2^2 x_3 - P_0^2 n_3^2 k_2 k_3 x_2$$

$$c_8^{(2)} = -P_2^2 n_1^2 k_3^2 - P_2^2 n_2^2 k_3^2 + 2 P_2 P_3 n_1^2 k_2 k_3 + 2 P_2 P_3 n_2^2 k_2 k_3 - 2 P_2 P_0 n_1^2 k_2 k_3 x_3 + 2 P_2 P_0 n_1^2 k_3^2 x_2 - 2 P_2 P_0 n_2^2 k_2 k_3 x_3 + 2 P_2 P_0 n_2^2 k_3^2 x_2 - P_3^2 n_1^2 k_2^2 - P_3^2 n_2^2 k_2^2 - P_3^2 n_2^2 k_3^2 + 2 P_3^2 n_2 n_3 k_2 k_3 - P_3^2 n_3^2 k_2^2 + 2 P_3 P_0 n_1^2 k_2^2 x_3 - 2 P_3 P_0 n_1^2 k_2 k_3 x_2 + 2 P_3 P_0 n_2^2 k_2^2 x_3 - 2 P_3 P_0 n_2^2 k_2 k_3 x_2 + 2 P_3 P_0 n_2^2 k_3^2 x_3 - 4 P_3 P_0 n_2 n_3 k_2 k_3 x_3 + 2 P_3 P_0 n_3^2 k_2^2 x_3 - P_0^2 n_1^2 k_2^2 x_3^2 + 2 P_0^2 n_1^2 k_2 k_3 x_2 x_3 - P_0^2 n_1^2 k_3^2 x_2^2 - P_0^2 n_2^2 k_2^2 x_3^2 + 2 P_0^2 n_2^2 k_2 k_3 x_2 x_3 + P_0^2 n_2^2 k_3^2 d^2 - P_0^2 n_2^2 k_3^2 x_2^2 - P_0^2 n_2^2 k_3^2 x_3^2 - 2 P_0^2 n_2 n_3 k_2 k_3 d^2 + 2 P_0^2 n_2 n_3 k_2 k_3 x_3^2 + P_0^2 n_3^2 k_2^2 d^2 - P_0^2 n_3^2 k_2^2 x_3^2$$

$$c_1^{(3)} = -2 P_0 n_1^3 n_3 k_1 k_3 r + P_0 n_1^2 n_3^2 k_1^2 r - 2 P_0 n_1 n_2^2 n_3 k_1 k_3 r - 2 P_0 n_1 n_3^3 k_1 k_3 r + P_0 n_2^2 n_3^2 k_1^2 r + P_0 n_3^4 k_1^2 r + n_1^2 k_3^2$$

$$c_2^{(3)} = 2 P_1 P_0 n_1 n_2 n_3 k_1 k_3 r + P_1 n_2 k_3^2 - 2 P_2 P_0 n_1^2 n_3 k_1 k_3 r + P_2 n_1 k_3^2 -$$

$$2 P_3 P_0 n_1^2 n_2 k_1 k_3 r - 2 P_3 n_2 k_1 k_3 + 2 P_0^2 n_1^2 n_2 k_1 k_3 r x_3 + 2 P_0^2 n_1^2 n_3 k_1 k_3 r x_2 - 2 P_0^2 n_1 n_2 n_3 k_1 k_3 r x_1 - P_0 n_1 k_3^2 x_2 + P_0 n_2 k_3^2 x_1$$

$$c_3^{(3)} = 2 P_1 P_0 n_1^3 k_1 k_3 r - P_1 P_0 n_1^2 n_3 k_1^2 r + 2 P_1 P_0 n_1 n_3^2 k_1 k_3 r - P_1 P_0 n_2^2 n_3 k_1^2 r - P_1 P_0 n_3^3 k_1^2 r + P_1 n_3 k_3^2 + 4 P_2 P_0 n_1^2 n_2 k_1 k_3 r - P_3 P_0 n_1^3 k_1^2 r - P_3 P_0 n_1 n_2^2 k_1^2 r - P_3 P_0 n_1 n_3^2 k_1^2 r + P_3 n_1 k_3^2 - 2 P_3 n_3 k_1 k_3 + P_0^2 n_1^3 k_1^2 r x_3 - 2 P_0^2 n_1^3 k_1 k_3 r x_1 - 4 P_0^2 n_1^2 n_2 k_1 k_3 r x_2 + P_0^2 n_1^2 n_3 k_1^2 r x_1 + P_0^2 n_1 n_2^2 k_1^2 r x_3 + P_0^2 n_1 n_3^2 k_1^2 r x_3 - 2 P_0^2 n_1 n_3^2 k_1 k_3 r x_1 + P_0^2 n_2^2 n_3 k_1^2 r x_1 + P_0^2 n_3^3 k_1^2 r x_1 - P_0 n_1 k_3^2 x_3 + P_0 n_3 k_3^2 x_1$$

$$c_4^{(3)} = -P_2 P_0 n_1^2 n_3 k_1^2 r + 2 P_2 P_0 n_1 n_2^2 k_1 k_3 r + 2 P_2 P_0 n_1 n_3^2 k_1 k_3 r - P_2 P_0 n_2^2 n_3 k_1^2 r - P_2 P_0 n_3^3 k_1^2 r - P_3 P_0 n_1^2 n_2 k_1^2 r - P_3 P_0 n_2^2 n_3 k_1^2 r + P_0^2 n_1^2 n_2 k_1^2 r x_3 + P_0^2 n_1^2 n_3 k_1^2 r x_2 - 2 P_0^2 n_1 n_2^2 k_1 k_3 r x_2 - 2 P_0^2 n_1 n_3^2 k_1 k_3 r x_2 + P_0^2 n_3^3 k_1^2 r x_3 + P_0^2 n_2^2 n_3 k_1^2 r x_2 + P_0^2 n_2 n_3^2 k_1^2 r x_3 + P_0^2 n_3^3 k_1^2 r x_2$$

$$c_5^{(3)} = 2 P_1 P_0 n_1^2 k_3^2 + 2 P_2 P_0 n_1 n_2 k_3^2 - 2 P_3 P_0 n_1^2 k_1 k_3 + 2 P_3 P_0 n_1 n_3 k_3^2 - 2 P_3 P_0 n_2^2 k_1 k_3 - 2 P_3 P_0 n_3^2 k_1 k_3 + P_0^2 n_1^2 k_1 k_3 x_3 - P_0^2 n_1^2 k_3^2 x_1 - 2 P_0^2 n_1 n_2 k_3^2 x_2 - 2 P_0^2 n_1 n_3 k_3^2 x_3 + P_0^2 n_2^2 k_1 k_3 x_3 + P_0^2 n_2^2 k_3^2 x_1 + P_0^2 n_3^2 k_1 k_3 x_3 + P_0^2 n_3^2 k_3^2 x_1$$

$$c_6^{(3)} = 0$$

$$c_7^{(3)} = -2 P_1 P_0 n_1^2 k_1 k_3 - 2 P_2 P_0 n_1 n_2 k_1 k_3 + 2 P_3 P_0 n_1^2 k_1^2 - 2 P_3 P_0 n_1 n_3 k_1 k_3 + 2 P_3 P_0 n_2^2 k_1^2 + 2 P_3 P_0 n_3^2 k_1^2 - P_0^2 n_1^2 k_1^2 x_3 + P_0^2 n_1^2 k_1 k_3 x_1 + 2 P_0^2 n_1 n_2 k_1 k_3 x_2 + 2 P_0^2 n_1 n_3 k_1 k_3 x_3 - P_0^2 n_2^2 k_1^2 x_3 - P_0^2 n_2^2 k_1 k_3 x_1 - P_0^2 n_3^2 k_1^2 x_3 - P_0^2 n_3^2 k_1 k_3 x_1$$

$$c_8^{(3)} = 2 P_1 P_3 n_1^2 k_1 k_3 - 2 P_1 P_0 n_1^2 k_1 k_3 x_3 + 2 P_2 P_3 n_1 n_2 k_1 k_3 - 2 P_2 P_0 n_1 n_2 k_1 k_3 x_3 - P_3^2 n_1^2 k_1^2 + 2 P_3^2 n_1 n_3 k_1 k_3 - P_3^2 n_2^2 k_1^2 - P_3^2 n_3^2 k_1^2 + 2 P_3 P_0 n_1^2 k_1^2 x_3 - 2 P_3 P_0 n_1^2 k_1 k_3 x_1 - 2 P_3 P_0 n_1 n_2 k_1 k_3 x_2 - 4 P_3 P_0 n_1 n_3 k_1 k_3 x_3 + 2 P_3 P_0 n_2^2 k_1^2 x_3 + 2 P_3 P_0 n_3^2 k_1^2 x_3 - P_0^2 n_1^2 k_1^2 x_3^2 + 2 P_0^2 n_1^2 k_1 k_3 x_1 x_3 + 2 P_0^2 n_1 n_2 k_1 k_3 x_2 x_3 - 2 P_0^2 n_1 n_3 k_1 k_3 d^2 + 2 P_0^2 n_1 n_3 k_1 k_3 x_3^2 - P_0^2 n_2^2 k_1^2 x_3^2 + P_0^2 n_3^2 k_1^2 d^2 - P_0^2 n_3^2 k_1^2 x_3^2$$

Generic Offset Polynomial for Example 3.25 (page 80)

$$g(d, x_1, x_2, x_3) = -16 x_1^2 x_2^{10} x_3^2 + 16 x_1^2 x_2^{10} d^2 - 48 x_1^2 x_2^8 x_3^4 + 128 x_1^2 x_2^8 x_3^2 d^2 - 80 x_1^2 x_2^8 d^4 - 48 x_1^2 x_2^6 x_3^6 + 240 x_1^2 x_2^6 x_3^4 d^2 - 352 x_1^2 x_2^6 x_3^2 d^4 + 160 x_1^2 x_2^6 d^6 - 16 x_1^2 x_2^4 x_3^8 + 160 x_1^2 x_2^4 x_3^6 d^2 - 432 x_1^2 x_2^4 x_3^4 d^4 + 448 x_1^2 x_2^4 x_3^2 d^6 - 160 x_1^2 x_2^4 d^8 + 32 x_1^2 x_2^2 x_3^8 d^2 - 176 x_1^2 x_2^2 x_3^6 d^4 + 336 x_1^2 x_2^2 x_3^4 d^6 - 272 x_1^2 x_2^2 x_3^2 d^8 + 80 x_1^2 x_2^2 d^{10} - 16 x_1^2 x_2^8 d^4 + 64 x_1^2 x_2^6 d^6 - 96 x_1^2 x_2^4 d^8 + 64 x_1^2 x_2^3 d^{10} - 16 x_1^2 d^{12} - 16 x_2^{12} x_3^2 + 16 x_2^{12} d^2 - 64 x_2^{10} x_3^4 + 160 x_2^{10} x_3^2 d^2 - 96 x_2^{10} d^4 - 96 x_2^8 x_3^6 + 416 x_2^8 x_3^4 d^2 - 560 x_2^8 x_3^2 d^4 + 240 x_2^8 d^6 - 64 x_2^6 x_3^8 + 448 x_2^6 x_3^6 d^2 - 1024 x_2^6 x_3^4 d^4 + 960 x_2^6 x_3^2 d^6 - 320 x_2^6 d^8 - 16 x_2^4 x_3^{10} + 208 x_2^4 x_3^8 d^2 - 768 x_2^4 x_3^6 d^4 + 1216 x_2^4 x_3^4 d^6 - 880 x_2^4 x_3^2 d^8 + 240 x_2^4 d^{10} + 32 x_2^2 x_3^{10} d^2 - 224 x_2^2 x_3^8 d^4 + 576 x_2^2 x_3^6 d^6 - 704 x_2^2 x_3^4 d^8 + 416 x_2^2 x_3^2 d^{10} - 96 x_2^2 d^{12} - 16 x_3^{10} d^4 + 80 x_3^8 d^6 - 160 x_3^6 d^8 + 160 x_3^4 d^{10} - 80 x_3^2 d^{12} + 16 d^{14} + 32 x_1^4 x_2^8 x_3 + 744 x_1^4 x_2^6 x_3^3 - 808 x_1^4 x_2^6 x_3 d^2 - 120 x_1^4 x_2^4 x_3^5 - 1080 x_1^4 x_2^4 x_3^3 d^2 + 1232 x_1^4 x_2^4 x_3 d^4 + 32 x_1^4 x_2^2 x_3^7 + 408 x_1^4 x_2^2 x_3^5 d^2 - 272 x_1^4 x_2^2 x_3^3 d^4 - 168 x_1^4 x_2^2 x_3 d^6 +$$

$$\begin{aligned}
& 32x_1^4x_3^7d^2 - 352x_1^4x_3^5d^4 + 608x_1^4x_3^3d^6 - 288x_1^4x_3d^8 + 32x_1^2x_2^{10}x_3 + 968x_1^2x_2^8x_3^3 - \\
& 1032x_1^2x_2^8x_3d^2 + 720x_1^2x_2^6x_3^5 - 3176x_1^2x_2^6x_3^3d^2 + 2488x_1^2x_2^6x_3d^4 - 184x_1^2x_2^4x_3^7 - \\
& 392x_1^2x_2^4x_3^5d^2 + 2168x_1^2x_2^4x_3^3d^4 - 1592x_1^2x_2^4x_3d^6 + 32x_1^2x_2^2x_3^9 + 632x_1^2x_2^2x_3^7d^2 - \\
& 1672x_1^2x_2^2x_3^5d^4 + 1320x_1^2x_2^2x_3^3d^6 - 312x_1^2x_2^2x_3d^8 + 32x_1^2x_3^9d^2 - 512x_1^2x_3^7d^4 + \\
& 1344x_1^2x_3^5d^6 - 1280x_1^2x_3^3d^8 + 416x_1^2x_3d^{10} + 160x_2^{10}x_3^3 - 160x_2^{10}x_3d^2 + 224x_2^8x_3^5 - \\
& 736x_2^8x_3^3d^2 + 512x_2^8x_3d^4 - 32x_2^6x_3^7 - 256x_2^6x_3^5d^2 + 736x_2^6x_3^3d^4 - 448x_2^6x_3d^6 - \\
& 96x_2^4x_3^9 + 544x_2^4x_3^7d^2 - 928x_2^4x_3^5d^4 + 608x_2^4x_3^3d^6 - 128x_2^4x_3d^8 + 224x_2^2x_3^9d^2 - \\
& 1024x_2^2x_3^7d^4 + 1728x_2^2x_3^5d^6 - 1280x_2^2x_3^3d^8 + 352x_2^2x_3d^{10} - 128x_3^9d^4 + 512x_3^7d^6 - \\
& 768x_3^5d^8 + 512x_3^3d^{10} - 128x_3d^{12} - 16x_1^6x_2^6 - 2073x_1^6x_2^4x_3^2 + 873x_1^6x_2^4d^2 + 384x_1^6x_2^2x_3^4 - \\
& 324x_1^6x_2^2x_3^2d^2 + 2052x_1^6x_2^2d^4 - 16x_1^6x_3^6 + 504x_1^6x_3^4d^2 - 1728x_1^6x_3^2d^4 + 216x_1^6d^6 - \\
& 16x_1^4x_2^8 - 2797x_1^4x_2^6x_3^2 + 1277x_1^4x_2^6d^2 - 2529x_1^4x_2^4x_3^4 + 2602x_1^4x_2^4x_3^2d^2 + 2615x_1^4x_2^4d^4 + \\
& 560x_1^4x_2^2x_3^6 + 980x_1^4x_2^2x_3^4d^2 + 552x_1^4x_2^2x_3^2d^4 - 3372x_1^4x_2^2d^6 - 16x_1^4x_3^8 + 744x_1^4x_3^6d^2 - \\
& 3992x_1^4x_3^4d^4 + 3768x_1^4x_3^2d^6 - 504x_1^4d^8 - 620x_1^2x_2^8x_3^2 + 364x_1^2x_2^8d^2 - 1060x_1^2x_2^6x_3^4 + \\
& 252x_1^2x_2^6x_3^2d^2 + 1256x_1^2x_2^6d^4 - 824x_1^2x_2^4x_3^6 + 1436x_1^2x_2^4x_3^4d^2 + 2448x_1^2x_2^4x_3^2d^4 - \\
& 3252x_1^2x_2^4d^6 + 192x_1^2x_2^2x_3^8 + 1952x_1^2x_2^2x_3^6d^2 - 4032x_1^2x_2^2x_3^4d^4 + 608x_1^2x_2^2x_3^2d^6 + \\
& 1280x_1^2x_2^2d^8 + 224x_1^2x_3^8d^2 - 2432x_1^2x_3^6d^4 + 4544x_1^2x_3^4d^6 - 2688x_1^2x_3^2d^8 + 352x_1^2d^{10} + \\
& 8x_2^{10}x_3^3 - 8x_2^{10}d^2 - 480x_2^8x_3^4 + 296x_2^8x_3^2d^2 + 184x_2^8d^4 + 296x_2^6x_3^6 - 8x_2^6x_3^4d^2 + 152x_2^6x_3^2d^4 - \\
& 440x_2^6d^6 - 240x_2^4x_3^8 + 104x_2^4x_3^6d^2 + 424x_2^4x_3^4d^4 - 584x_2^4x_3^2d^6 + 296x_2^4d^8 + 672x_2^2x_3^8d^2 - \\
& 1792x_2^2x_3^6d^4 + 1600x_2^2x_3^4d^6 - 512x_2^2x_3^2d^8 + 32x_2^2d^{10} - 448x_3^8d^4 + 1408x_3^6d^6 - 1536x_3^4d^8 + \\
& 640x_3^2d^{10} - 64d^{12} + 2106x_1^8x_2^2x_3 - 216x_1^8x_3^3 + 1944x_1^8x_3d^2 + 2946x_1^6x_2^4x_3 + 3282x_1^6x_2^2x_3^3 + \\
& 54x_1^6x_2^2x_3d^2 - 312x_1^6x_3^5 + 4176x_1^6x_3^3d^2 - 5400x_1^6x_3d^4 + 760x_1^4x_2^6x_3 + 800x_1^4x_2^4x_3^3 + \\
& 56x_1^4x_2^4x_3d^2 + 1744x_1^4x_2^2x_3^5 + 2048x_1^4x_2^2x_3^3d^2 - 3696x_1^4x_2^2x_3d^4 - 96x_1^4x_3^7 + 2784x_1^4x_3^5d^2 - \\
& 8992x_1^4x_3^3d^4 + 5280x_1^4x_3d^6 - 16x_1^2x_2^8x_3 + 1362x_1^2x_2^6x_3^3 - 546x_1^2x_2^6x_3d^2 - 1560x_1^2x_2^4x_3^5 + \\
& 2880x_1^2x_2^4x_3^3d^2 - 2472x_1^2x_2^4x_3d^4 + 480x_1^2x_2^2x_3^7 + 2120x_1^2x_2^2x_3^5d^2 - 3408x_1^2x_2^2x_3^3d^4 + \\
& 1192x_1^2x_2^2x_3d^6 + 672x_1^2x_3^7d^2 - 5088x_1^2x_3^5d^4 + 6624x_1^2x_3^3d^6 - 2208x_1^2x_3d^8 - 72x_2^8x_3^3 + \\
& 72x_2^8x_3d^2 + 440x_2^6x_3^5 + 592x_2^6x_3^3d^2 - 1032x_2^6x_3d^4 - 320x_2^4x_3^7 - 864x_2^4x_3^5d^2 + 896x_2^4x_3^3d^4 + \\
& 288x_2^4x_3d^6 + 1120x_2^2x_3^7d^2 - 1440x_2^2x_3^5d^4 + 32x_2^2x_3^3d^6 + 288x_2^2x_3d^8 - 896x_3^7d^4 + \\
& 2176x_3^5d^6 - 1664x_3^3d^8 + 384x_3d^{10} - 729x_1^{10} - 1053x_1^8x_2^2 - 1377x_1^8x_2^2 + 2673x_1^8d^2 - \\
& 300x_1^6x_2^4 + 684x_1^6x_2^2x_3^2 + 3132x_1^6x_2^2d^2 - 888x_1^6x_3^4 + 5616x_1^6x_3^2d^2 - 3672x_1^6d^4 + 8x_1^4x_2^6 - \\
& 1500x_1^4x_2^4x_3^2 + 1072x_1^4x_2^4d^2 + 2232x_1^4x_2^2x_3^4 + 456x_1^4x_2^2x_3^2d^2 - 2576x_1^4x_2^2d^4 - 240x_1^4x_3^6 + \\
& 4384x_1^4x_3^4d^2 - 8272x_1^4x_3^2d^4 + 2336x_1^4d^6 + 276x_1^2x_2^6x_3^2 - 164x_1^2x_2^6d^2 - 1336x_1^2x_2^4x_3^4 + \\
& 1048x_1^2x_2^4x_3^2d^2 - 1024x_1^2x_2^4d^4 + 640x_1^2x_2^2x_3^6 + 480x_1^2x_2^2x_3^4d^2 - 112x_1^2x_2^2x_3^2d^4 + \\
& 272x_1^2x_2^2d^6 + 1120x_1^2x_3^6d^2 - 5760x_1^2x_3^4d^4 + 5088x_1^2x_3^2d^6 - 704x_1^2d^8 - 1x_2^8x_3^2 + 1x_2^8d^2 + \\
& 184x_2^6x_3^4 - 112x_2^6x_3^2d^2 - 72x_2^6d^4 - 240x_2^4x_3^6 - 896x_2^4x_3^4d^2 + 656x_2^4x_3^2d^4 + 480x_2^4d^6 + \\
& 1120x_2^2x_3^6d^2 - 480x_2^2x_3^4d^4 - 864x_2^2x_3^2d^6 + 224x_2^2d^8 - 1120x_3^6d^4 + 2080x_3^4d^6 - 1056x_3^2d^8 + \\
& 96d^{10} - 648x_1^8x_3 + 834x_1^6x_2^2x_3 - 968x_1^6x_3^3 + 2664x_1^6x_3d^2 - 336x_1^4x_2^4x_3 + 1352x_1^4x_2^2x_3^3 - \\
& 1408x_1^4x_2^2x_3d^2 - 320x_1^4x_3^5 + 3456x_1^4x_3^3d^2 - 3776x_1^4x_3d^4 + 2x_1^2x_2^6x_3 - 464x_1^2x_2^4x_3^3 + \\
& 176x_1^2x_2^4x_3d^2 + 480x_1^2x_2^2x_3^5 - 568x_1^2x_2^2x_3^3d^2 + 1176x_1^2x_2^2x_3d^4 + 1120x_1^2x_3^5d^2 - \\
& 3776x_1^2x_3^3d^4 + 2144x_1^2x_3d^6 + 8x_2^6x_3^3 - 8x_2^6x_3d^2 - 96x_2^4x_3^5 - 256x_2^4x_3^3d^2 + 352x_2^4x_3d^4 + \\
& 672x_2^2x_3^5d^2 - 64x_2^2x_3^3d^4 - 608x_2^2x_3d^6 - 896x_3^5d^4 + 1280x_3^3d^6 - 384x_3d^8 - 216x_1^8 + \\
& 132x_1^6x_2^2 - 456x_1^6x_3^2 + 720x_1^6d^2 - 1x_1^4x_2^4 + 376x_1^4x_2^2x_3^2 - 388x_1^4x_2^2d^2 - 240x_1^4x_3^4 + \\
& 1416x_1^4x_3^2d^2 - 856x_1^4d^4 - 32x_1^2x_2^4x_3^2 + 20x_1^2x_2^4d^2 + 192x_1^2x_2^2x_3^4 - 288x_1^2x_2^2x_3^2d^2 +
\end{aligned}$$

$$\begin{aligned}
& 416 x_1^2 x_2^2 d^4 + 672 x_1^2 x_3^4 d^2 - 1472 x_1^2 x_3^2 d^4 + 416 x_1^2 d^6 - 16 x_2^4 x_3^4 + 8 x_2^4 x_3^2 d^2 + 8 x_2^4 d^4 + \\
& 224 x_2^2 x_3^4 d^2 - 64 x_2^2 x_3^2 d^4 - 160 x_2^2 d^6 - 448 x_3^4 d^4 + 512 x_3^2 d^6 - 64 d^8 - 96 x_1^6 x_3 + 40 x_1^4 x_2^2 x_3 - \\
& 96 x_1^4 x_3^3 + 320 x_1^4 x_3 d^2 + 32 x_1^2 x_2^2 x_3^3 - 8 x_1^2 x_2^2 x_3 d^2 + 224 x_1^2 x_3^3 d^2 - 352 x_1^2 x_3 d^4 + 32 x_2^2 x_3^3 d^2 - \\
& 32 x_2^2 x_3 d^4 - 128 x_3^3 d^4 + 128 x_3 d^6 - 16 x_1^6 - 16 x_1^4 x_3^2 + 48 x_1^4 d^2 + 32 x_1^2 x_3^2 d^2 - 48 x_1^2 d^4 - \\
& 16 x_3^2 d^4 + 16 d^6.
\end{aligned}$$

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